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ON DEFINING EQUATIONS FOR THE JACOBIAN LOCUS IN GENUS FIVE[1]

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ABSTRACT. In the space of principally polarized abelian varieties of dimension 5, eight special theta relations can be chosen to define eight hypersurfaces whose intersection contains the Jacobian locus as a component.

Let $\mathfrak{S}_5$ be the Siegel generalized upper half plane of principally polarized abelian varieties of dimension 5. Let $\mathcal{J}_5$ be the Jacobian locus and $\mathcal{H}_5$ the hyperelliptic sublocus of $\mathcal{J}_5$. Then $\mathcal{H}_5 \subset \mathcal{J}_5 \subset \mathfrak{S}_5$ with respective dimensions 9, 12, and 15. From the Schottky-Jung-Farkas-Rauch theorem [**5**, p. 212] one can derive equations in thetanulls, the special theta relations, which hold for all $A \in \mathcal{J}_5$ but not for all $A \in \mathfrak{S}_5$. These special theta relations therefore define hypersurfaces in $\mathfrak{S}_5$, each of which contains $\mathcal{J}_5$. In this paper we show that the intersection of these hypersurfaces contains $\mathcal{J}_5$ as a component; that is, these special theta relations give, locally, defining equations for $\mathcal{J}_5$ in $\mathfrak{S}_5$.

A recent result of D. Mumford is an essential ingredient of the proof. Otherwise we use methods that go back to M. Noether [**4**]. We must presuppose some acquaintance with the classical theory of theta functions. As references we give [**5**], [**2**, Chapter 7] and also [**1**] for a very brief resumé. We will briefly introduce some of the necessary ideas for dimension 5, hoping they are already somewhat familiar to the reader. Our notation will follow [**2**].

Let $(\pi iE, B)$ be a 5×10 period matrix for an abelian variety $A \in \mathfrak{S}_5$. For $u \in C^5$ let $\theta[\varepsilon](u; B)$ be a first order theta function with theta characteristic (Th. Char.) $[\varepsilon]$. $[\varepsilon]$ is called an *even* (*odd*) Th. Char. if this function is even (odd), and this is denoted $|\varepsilon| = 1$ ($|\varepsilon| = -1$). If $|\varepsilon| = 1$ we call $\theta[\varepsilon](0; B)$ a *thetanull* and write it simply as $\theta[\varepsilon]$. A set of even theta characteristics $\{[\varepsilon_i]\}$ will be called *azygetic* if $|\varepsilon_i\varepsilon_j\varepsilon_k| = -1$ for 3 different Th. Char.'s and if the sum of an even number is not the zero period characteristic (Per. Char.), except possibly for the sum of all the $[\varepsilon_i]$. For dimension 5 an azygetic set of eleven even Th. Char.'s $\{[\varepsilon_i]\}$ together with $[n]$ $(= [\sum_{i=1}^{11}\varepsilon_i])$ will be called a hyperelliptic fundamental system of theta characteristics (H.F.S. of Th. Char.'s). In this case $|n| = -1$ and $|n\varepsilon_i\varepsilon_j| = 1$ if $i \neq j$. If we denote the Per. Char.

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$(n\varepsilon_i)$ by (a_i) then $[\varepsilon_i] = [na_i]$ and $[n\varepsilon_i\varepsilon_j] = [na_ia_j]$. Write $[n\Sigma^s a]$ for a Th. Char. where $\Sigma^s a$ stands for a sum of s different a's. Then $[n\Sigma^s a]$ is even if $s = 1, 2, 5, 6, 9, 10$ and odd if $s = 0, 3, 4, 7, 8, 11$.

Let W_5 be a hyperelliptic Riemann surface of genus 5. By choosing a suitable canonical homology basis on W_5 [**2**, p. 445] one can find a H.F.S. of Th. Char.'s so that $\theta[na_i] = \theta[na_ia_j] = 0$ for all i, j and $\theta[n\Sigma^5 a] \neq 0$ for all Th. Char. of this form. (Since $(\Sigma_{i=1}^{11} a_i) = (0)$ it follows that $\theta[n\Sigma^6 a] \neq 0$ also.)

General theta relations are homogeneous polynomials in the thetanulls which are zero for all $A \in \mathfrak{S}_5$. We now describe those general theta relations for dimension 5 which we will need [**4**].

Suppose $[\varepsilon_1], \ldots, [\varepsilon_6]$ is an azygetic set of six even Th. Char.'s and G_4 is a group of four Per. Char.'s so that (i) $|\varepsilon_i\sigma| = 1$ for $i = 1, 2, \ldots, 6$ and all $(\sigma) \in G_4$, and (ii) $(\Sigma_{i=1}^{6} \varepsilon_i) \in G_4$. Let

$$q_i = \prod_{(\sigma) \in G_4} \theta[\varepsilon_i\sigma].$$

Then

$$(\mathrm{GR})_1 \qquad \sum_{i=1}^{6} \pm q_i = 0.$$

Suppose $[\varepsilon_1], \ldots, [\varepsilon_4]$ is an azygetic set of four even Th. Char.'s and G_8 is a group of 8 Per. Char.'s so that $|\varepsilon_i\sigma| = 1$ for $i = 1, \ldots, 4$ and for all $(\sigma) \in G_8$. Let G_4 be a subgroup of G_8 of order four and let $(\tau) \in G_8 - G_4$. Let

$$q_i = \prod_{(\sigma) \in G_4} \theta[\varepsilon_i\sigma] \quad \text{and} \quad q_i' = \prod_{(\sigma) \in G_4} \theta[\varepsilon_i\sigma\tau].$$

Then

$$(\mathrm{GR})_2 \qquad \sum_{i=1}^{4} (\pm q_i \pm q_i') = 0.$$

As an application of $(\mathrm{GR})_2$ for a H.F.S. of Th. Char.'s let $[\varepsilon_i] = [na_i]$, $i = 1, 2, 3, 4$. Let $(\tau) = (a_5)$ and

$$G_4 = \{(0), (a_6a_7a_8a_9), (a_6a_7a_{10}a_{11}), (a_8a_9a_{10}a_{11})\}.$$

Then

$$q_i = \theta[na_i]\theta[na_ia_6a_7a_8a_9]\theta[na_ia_6a_7a_{10}a_{11}]\theta[na_ia_8a_9a_{10}a_{11}]$$

and

$$q_i' = \theta[na_ia_5]\theta[na_ia_5a_6a_7a_8a_9]\theta[na_ia_5a_6a_7a_{10}a_{11}]\theta[na_ia_5a_8a_9a_{10}a_{11}].$$

Now by an arbitrary permutation of the digits from 1 to 11 we obtain the following lemma.

LEMMA 1. *Suppose for* 5 *distinct integers* i, j, k, l, m *from* 1 *to* 11 *seven of the eight thetanulls with the following Th. Char.'s are zero*: $[na_i]$, $[na_j]$, $[na_k]$, $[na_l]$, $[na_ia_m]$, $[na_ja_m]$, $[na_ka_m]$, $[na_la_m]$. *Suppose all* $\theta[n\Sigma^5 a] \neq 0$. *Then the thetanull with the eighth Th. Char. is also zero.*

Using a recent result of D. Mumford we now give local defining equations for $\mathcal{H}_5$ in $\mathfrak{S}_5$.

LEMMA 2. *Suppose $A \in \mathfrak{S}_5$ is near $\mathcal{H}_5$ so that $\theta[n\Sigma^5 a] \neq 0$ for all such Th. Char.'s with respect to a suitable H.F.S. of Th. Char.'s. Suppose $\theta[na_i] = 0$ for $i = 1, 2, \ldots, 11$. Then $A \in \mathcal{H}_5$.*

PROOF. We first systematically apply the general theta relations to show that $\theta[na_i a_j] = 0$ for $1 \leqslant i < j \leqslant 11$. First apply $(\mathrm{GR})_1$ to $[na_1]$, $[na_2], \ldots, [na_5]$, $[na_6 a_7]$. Let

$$G_4 = \{(0), (a_8 a_9 a_{10} a_{11}), (a_6 a_7 a_8 a_9), (a_6 a_7 a_{10} a_{11})\}.$$

Then

$$q_6 = \theta[na_6 a_7]\theta[na_8 a_9]\theta[na_{10} a_{11}]\theta[na_6 a_7 a_8 a_9 a_{10} a_{11}] = 0.$$

We conclude that for any 6 distinct integers from 1 to 11,

$$\theta[na_i a_j]\theta[na_k a_l]\theta[na_m a_r] = 0.$$

Suppose $\theta[na_1 a_2] \neq 0$. Then $\theta[na_i a_j]\theta[na_k a_l] = 0$ for $3 \leqslant i < j < k < l \leqslant 11$. Suppose now, in addition, that $\theta[na_3 a_4] \neq 0$. Then $\theta[na_k a_l] = 0$ for $5 \leqslant k < l \leqslant 11$. Now apply Lemma 1 to $[na_3]$, $[na_5]$, $[na_6]$, $[na_7]$, $[na_3 a_8]$, $[na_5 a_8]$, $[na_6 a_8]$, $[na_7 a_8]$ to conclude that $\theta[na_3 a_8] = 0$. Thus $\theta[na_3 a_k] = \theta[na_4 a_k] = 0$ for $5 \leqslant k \leqslant 11$.

Now apply Lemma 1 to $[na_3]$, $[na_5]$, $[na_6]$, $[na_7]$, $[na_3 a_4]$, $[na_4 a_5]$, $[na_4 a_6]$, $[na_4 a_7]$ to conclude that $\theta[na_3 a_4] = 0$ after all. Thus the hypothesis $\theta[na_1 a_2] \neq 0$ implies $\theta[na_k a_l] = 0$ for $3 \leqslant k < l \leqslant 11$. By an entirely analogous argument we conclude that $\theta[na_1 a_2]$ itself must be zero.

Thus the theta functions for A have vanishing properties which mimic those of a hyperelliptic Jacobian. Mumford's theorem [**3**] states that such an A must be in $\mathcal{H}_5$; that is, $\mathcal{H}_5$ is characterized by the vanishings of its theta functions with half-integer Th. Char.'s. Q.E.D.

We now consider the special theta relations for genus 5, that is, equations in the thetanulls which hold on $\mathcal{J}_5$ but not on $\mathfrak{S}_5$.

Let $[\varepsilon_1], \ldots, [\varepsilon_4]$ be an azygetic set of four even Th. Char.'s. Let G_8 be a group of eight Per. Char.'s so that $|\varepsilon_i \sigma| = 1$ for all i and all $(\sigma) \in G_8$. Let

$$r_i = \prod_{(\sigma) \in G_8} \theta[\varepsilon_i \sigma].$$

Then

$$(\mathrm{SR}) \qquad \sum_{i=1}^{4} \pm \sqrt{r_i} = 0$$

or

$$\left(\sum r_i^2 - 2\sum r_i r_j\right)^2 = 64 r_1 r_2 r_3 r_4.$$

We now show that eight special theta relations give local defining equations for $\mathcal{J}_5$ in $\mathfrak{S}_5$. Let $A_0 \in \mathcal{H}_5$ and let A $(\varepsilon \mathfrak{S}_5)$ be near A_0 so that $\theta[n\Sigma^5 a] \neq 0$ for a suitable H.F.S. of Th. Char. Near A_0, $\mathcal{H}_5$ is defined by eleven equations $\theta[na_i] = 0$,

$i = 1, 2, \ldots, 11$. We now write eleven equations locally defining $\mathcal{H}_5$ near A_0 in a different manner.

Apply formula (SR) letting $[\varepsilon_i]$ be $[na_1]$, $[na_2]$, $[na_3]$, $[na_4]$ and letting

$$G_8 = \langle (a_5 a_6 a_7 a_8), (a_5 a_6 a_9 a_{10}), (a_5 a_7 a_9 a_{11}) \rangle.$$

Then r_i is of the form $\theta[na_i]\Pi^7\theta[n\Sigma^5 a]$, where the last 7 factors of the products are not zero. Let $(\mathrm{SR})_{1234}$ be the special relation $\Sigma_{i=1}^4 \pm \sqrt{r_i} = 0$. Let $(\mathrm{SR})_{123k}$ be the special relation where 4 and k are transposed, $k = 5, 6, \ldots, 11$. Then the eleven equations

$$\theta[na_1] = \theta[na_2] = \theta[na_3] = 0$$

and $(\mathrm{SR})_{1234}$, $(\mathrm{SR})_{1235, \ldots,}(\mathrm{SR})_{12311}$ are equivalent to

$$\theta[na_i] = 0, \quad i = 1, \ldots, 11, \quad \text{near } A_0.$$

Consequently, this new set of eleven equations locally defines $\mathcal{H}_5$ in $\mathcal{S}_5$.

The eight special theta relations $(\mathrm{SR})_{123k}$ define eight hypersurfaces in $\mathcal{S}_5$ each containing $\mathcal{J}_5$. Let V be an irreducible component of the intersection of these eight hypersurfaces which contains $\mathcal{J}_5$. The equations $\theta[na_i] = 0$, $i = 1, 2, 3$, determine three hypersurfaces whose intersection with V equals $\mathcal{H}_5$ by Lemma 2. Consequently, V has codimension $\geqslant 3$. It follows that $V = \mathcal{J}_5$ since $\mathcal{J}_5$ has codimension 3.

THEOREM. *In $\mathcal{S}_5$ eight special theta relations can be chosen which give local defining equations for $\mathcal{J}_5$.*

We conclude with some comments in the case of genus 6. Let $\mathcal{P}_7$ be the principally polarized abelian varieties of dimension 6 which are Pryms of Riemann surfaces of genus 7. Then $\mathcal{J}_6 \subset \mathcal{P}_7 \subset \mathcal{S}_6$ where the dimensions are 15, 18, and 21. One would like to show that the special theta relations for genus 6 are local defining equations for $\mathcal{J}_6$ in $\mathcal{S}_6$. The author has been unable to generalize the methods of this paper to this case. However, it can be shown that eight special relations *as functions on* $\mathcal{P}_7$ define $\mathcal{J}_6$ locally. This is accomplished by specializing eight special relations for genus 6 to eight special relations for genus 5 known to define $\mathcal{J}_5$ locally. The generalization of this latter type of result to arbitrary genus appears possible.

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