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A NOTE ON COUNTABLY NORMED NUCLEAR SPACES

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ABSTRACT. A modification of the Kōmura-Kōmura imbedding theorem is used to show that every countably normed nuclear space is isomorphic to a subspace of a nuclear Fréchet space with basis and a continuous norm. The space with basis can be chosen to be a quotient of (s).

1. Introduction. By the famous Kōmura-Kōmura imbedding theorem [**5**] every nuclear Fréchet space is isomorphic to a subspace of $(s)^{\mathbf{N}}$, where (s) is the space of rapidly decreasing sequences. As a corollary, every nuclear Fréchet space is isomorphic to a subspace of a nuclear Fréchet space with basis. Since $(s)^{\mathbf{N}}$ does not admit a continuous norm, we can ask to what extent this corollary holds for spaces with a continuous norm. We will show that a nuclear Fréchet space with a continuous norm is isomorphic to a subspace of a nuclear Fréchet space with basis and a continuous norm if (and only if) it is countably normed. (The concept of countably normedness was very important in constructing the first example of a nuclear Fréchet space without the bounded approximation property (see [**1**]).) Moreover, the space with basis can be chosen to be a quotient of (s). The proof is a modification of the standard proof of the Kōmura-Kōmura theorem.

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2. Countably normed spaces. Let E be a Fréchet space which admits a continuous norm. The topology of E can then be defined by an increasing sequence $(\|\cdot\|_k)$ of norms (the index set is $\mathbf{N} = \{1, 2, \ldots\}$). Let E_k denote E equipped with the norm $\|\cdot\|_k$ only and let $\hat{E}_k$ be the completion of E_k. The identity mapping $E_{k+1} \to E_k$ has a unique extension $\phi_k\colon \hat{E}_{k+1} \to \hat{E}_k$ and this latter mapping is called *canonical*. The space E is said to be *countably normed* if the system $(\|\cdot\|_k)$ can be chosen in such a way that each ϕ_k is injective.

To give an example of a countably normed space, assume that E has an absolute basis i.e. there is a sequence (x_n) in E such that every $x \in E$ has a unique absolutely converging expansion $x = \Sigma_n \xi_n x_n$, where (ξ_n) is a sequence of scalars. Then E is isomorphic to the Köthe sequence space

$$(1) \qquad K(a) = K(a_n^k) = \left\{ (\xi_n) \,|\, |(\xi_n)|_k = \sum_n |\xi_n| a_n^k < \infty \ \forall k \right\},$$

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where $a_n^k = \|x_n\|_k$ (cf. [6, 10.1]). The topology of $K(a)$ is defined by the norms $|\cdot|_k$. The completions $(K(a)_k)\hat{}$ can be isometrically identified with l_1 and then the canonical mapping $\phi_k\colon l_1 \to l_1$ is the diagonal transformation $(\xi_n)_n \mapsto ((a_n^k/a_n^{k+1})\xi_n)_n$ which is clearly injective. Therefore E is countably normed.

Consider now a nuclear Fréchet space E which admits a continuous norm. The topology of E can be defined by a sequence $(\|\cdot\|_k)$ of Hilbert norms, that is, $\|x\|_k = \langle x, x\rangle_k$, $x \in E$, where $\langle\cdot,\cdot\rangle_k$ is an inner product on E. The following result is due to Ed Dubinsky and the proof will be contained in [3].

THEOREM 1. *If a nuclear Fréchet space E is countably normed, then the topology of E can be defined by a sequence of Hilbert norms such that the canonical mappings* ϕ_k: $\hat{E}_{k+1} \to \hat{E}_k$ *are injective.*

Suppose finally that (x_n) is a basis of E. Since (x_n) is necessarily absolute [6, 10.2.1], E can be identified with a Köthe space $K(a)$. By the Grothendieck-Pietsch nuclearity criterion [6, 6.1.2], for every k there is l with $(a_n^k/a_n^l) \in l_1$. Conversely, if the matrix (a_n^k) with $0 < a_n^k \leq a_n^{k+1}$ satisfies this criterion, then the Köthe space $K(a)$ defined through (1) is a nuclear Fréchet space with a continuous norm and the sequence of coordinate vectors constitutes a basis. In particular, $(s) = K(n^k)$. The topology of such a nuclear Köthe space can also be defined by the sup-norms, $|(\xi_n)|_{k,\infty} = \sup_n |\xi_n| a_n^k$.

3. An imbedding theorem. We are now ready to prove the following characterization of countably normed nuclear spaces.

THEOREM 2. *Let E be a nuclear Fréchet space which admits a continuous norm. Then the following two conditions are equivalent*:

(i) *E is countably normed*,

(ii) *E is isomorphic to a subspace of a nuclear Köthe space which admits a continuous norm.*

Moreover, the Köthe space in (ii) *can be chosen to be a quotient of* (s).

PROOF. As explained in the introduction, a nuclear Köthe space with a continuous norm is countably normed. Since countably normedness is inherited by subspaces (e.g. [1, VI, 3.1.4]), the implication (ii) $\Rightarrow$ (i) is clear.

To prove (i) $\Rightarrow$ (ii) we choose a sequence $(\|\cdot\|_k)$ of Hilbert norms defining the topology of E such that each canonical mapping $\phi_k\colon \hat{E}_{k+1} \to \hat{E}_k$ is injective (Theorem 1). Let $U_k = \{x \in E \mid \|x\|_k \leq 1\}$ and identify $(\hat{E}_k)'$ with

$$E_k' = \left\{ f \in E' \,\middle|\, \|f\|_k' = \sup_{x \in U_k} |\langle x, f\rangle| < \infty \right\}.$$

Then $\phi_k'\colon E_k' \to E_{k+1}'$ is simply the inclusion mapping. As a Hilbert space, E_{k+1}' is reflexive. Using this and the fact that $\phi_k\colon E_{k+1} \to E_k$ is injective, one sees easily that $\phi_k'(E_k') = E_k'$ is dense in E_{k+1}'.

As in the standard proof of the Kōmura-Kōmura theorem (e.g. [6, 11.1.1]) we can construct in each E_k' a sequence $(f_n^{(k)})_n$ of functionals with the following properties:

$$U_k^\circ \subset \{f_n^{(k)} \mid n \in \mathbf{N}\}^{\circ\circ}, \tag{2}$$

$$\{n^l f_n^{(k)} \mid n \in \mathbf{N}\} \text{ is equicontinuous for every } l. \tag{3}$$

Now set $g_n^{(1)} = f_n^{(1)}$, $n \in \mathbf{N}$, and using the fact that E_1' is dense in every E_k' choose $g_n^{(k)} \in E_1'$, $k \geq 2$, $n \in \mathbf{N}$, with

$$\| f_n^{(k)} - g_n^{(k)} \|_k' < 2^{-n}. \tag{4}$$

In the construction of the desired Köthe space $K(a)$ we will use two indices k and n to enumerate the coordinate basis vectors. First, set

$$a_{kn}^l = 2^k n^{2l}, \qquad k, n \in \mathbf{N}, l > k. \tag{5}$$

Then choose $a_{kn}^k, a_{kn}^{k-1}, \ldots, a_{kn}^1$ so that

$$1 > a_{kn}^k \geq a_{kn}^{k-1} \geq \cdots \geq a_{kn}^1 > 0, \qquad k, n \in \mathbf{N}, \tag{6}$$

$$\frac{a_{kn}^{l+1}}{a_{kn}^{l+2}} \geq \frac{a_{kn}^l}{a_{kn}^{l+1}}, \qquad k, n \in \mathbf{N}, l \leq k, \tag{7}$$

$$a_{kn}^l \leq \frac{1}{\| g_n^{(k)} \|_l'}, \qquad k, n \in \mathbf{N}, l \leq k. \tag{8}$$

Note that (7) holds trivially for $l > k$. Consequently, if $K(a_{kn}^l) = K(a)$ is nuclear, then it is also isomorphic to a quotient space of (s) [**2**, Theorem 2.4]. But by (7), (5) and (6) for every $l \geq 2$,

$$\sum_{k=1}^\infty \sum_{n=1}^\infty \frac{a_{kn}^l}{a_{kn}^{l+1}} = \sum_{k=1}^{l-1} \sum_{n=1}^\infty \frac{a_{kn}^l}{a_{kn}^{l+1}} + \sum_{k=l}^\infty \sum_{n=1}^\infty \frac{a_{kn}^l}{a_{kn}^{l+1}} \leq \sum_{k=1}^{l-1} \sum_{n=1}^\infty \frac{a_{kn}^l}{a_{kn}^{l+1}} + \sum_{k=l}^\infty \sum_{n=1}^\infty \frac{a_{kn}^k}{a_{kn}^{k+1}}$$
$$< (l-1) \sum_{n=1}^\infty \frac{1}{n^2} + \sum_{k=l}^\infty \sum_{n=1}^\infty \frac{1}{2^k n^{2(k+1)}} < \infty.$$

To imbed E into $K(a)$ we set $Ax = (\langle x, g_n^{(k)} \rangle)_{k,n}$, $x \in E$. We have to show that $Ax \in K(a)$, $A\colon E \to K(a)$ is a continuous injection and that $A^{-1}\colon A(E) \to E$ is also continuous.

Fix $l \geq 2$. Applying (3) to the sequences $(f_n^{(k)})_n$, $k = 1, \ldots, l-1$, we can find an index $p \geq l$ and a constant C such that

$$\sup_{k<l,n} 2^k n^{2l} |\langle x, f_n^{(k)} \rangle| \leq C \|x\|_p, \qquad x \in E.$$

From (5), (8) and (4) we then get for every $x \in E$,

$$|Ax|_{l,\infty} = \sup_{k,n} a_{kn}^l |\langle x, g_n^{(k)} \rangle| \leq \sup_{k<l,n} a_{kn}^l |\langle x, g_n^{(k)} \rangle| + \sup_{k \geq l,n} a_{kn}^l |\langle x, g_n^{(k)} \rangle|$$
$$\leq \sup_{k<l,n} 2^k n^{2l} |\langle x, g_n^{(k)} \rangle| + \sup_{k \geq l,n} \frac{1}{\| g_n^{(k)} \|_l'} |\langle x, g_n^{(k)} \rangle|$$
$$\leq \sup_{k<l,n} 2^k n^{2l} \| g_n^{(k)} - f_n^{(k)} \|_k' \|x\|_k$$
$$+ \sup_{k<l,n} 2^k n^{2l} |\langle x, f_n^{(k)} \rangle| + \|x\|_l \leq C' \|x\|_p,$$

where $C' = \sup_n n^{2l} 2^{l-n} + C + 1 < \infty$.

Consequently, $Ax \in K(a)$ and $A\colon E \to K(a)$ is continuous. From (2) it follows that for every $x \in E$,

$$\|x\|_l = \sup_{f \in U_l^\circ} |\langle x, f \rangle| \leq \sup_n |\langle x, f_n^{(l)} \rangle| . \tag{9}$$

Further, since $a_{ln}^{l+1} > 1$,

$$\begin{aligned} \sup_n |\langle x, f_n^{(l)} \rangle| &\leq \sup_n |\langle x, f_n^{(l)} - g_n^{(l)} \rangle| + \sup_n |\langle x, g_n^{(l)} \rangle| \\ &\leq \sup_n \| f_n^{(l)} - g_n^{(l)} \|_l' \|x\|_l + \sup_{k,n} a_{kn}^{l+1} |\langle x, g_n^{(k)} \rangle| \\ &\leq \frac{1}{2}\|x\|_l + |Ax|_{l+1,\infty}. \end{aligned} \tag{10}$$

Thus, by (9) and (10) we have for every $x \in E$,

$$\|x\|_l \leq 2\,|Ax|_{l+1,\infty}.$$

Since l was arbitrary, this shows that A is injective and that $A^{-1}\colon A(E) \to E$ is continuous. □

Finally we remark that is is not possible to find a *single* nuclear Fréchet space with basis and a continuous norm containing all countably normed nuclear spaces as subspaces. In fact, it was shown in **[4]** that not even any countable collection of nuclear Fréchet spaces with basis and a continuous norm contains all such spaces as subspaces.

References

1. E. Dubinsky, *The structure of nuclear Fréchet spaces*, Lecture Notes in Math., vol. 720, Springer-Verlag, Berlin-Heidelberg-New York, 1979.
2. E. Dubinsky and W. Robinson, *Quotient spaces of (s) with basis*, Studia Math. **63** (1978), 267–281.
3. E. Dubinsky and D. Vogt, *Fréchet spaces with quotients failing the bounded approximation property* (to appear).
4. L. Holmström, *Universal classes of nuclear Köthe spaces with a continuous norm*, J. Functional Anal. **48** (1982), 12–19.
5. T. Kōmura and Y. Kōmura, *Über die Einbettung der nuklearen Räume in $(s)^A$*, Math. Ann. **162** (1966), 284–288.
6. A. Pietsch, *Nuclear locally convex spaces*, Springer-Verlag, Berlin-Heidelberg-New York, 1972.

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