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EVENTUAL DISCONJUGACY OF A LINEAR DIFFERENTIAL EQUATION

WILLIAM F. TRENCH

ABSTRACT. A sufficient condition is given for eventual disconjugacy of an nth order linear differential equation.

1. Introduction. If there is an interval $[a, \infty)$ on which no nontrivial solution of the scalar equation

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = 0 \tag{1}$$

has more than $n-1$ zeros, counting multiplicities, then (1) is said to be *eventually disconjugate*. From a special case of a theorem of Willett [**4**], (1) is eventually disconjugate if $p_1, \ldots, p_n \in C[0, \infty)$ and

$$\int^{\infty} t^{k-1}|p_k(t)|\,dt < \infty, \qquad 1 \leq k \leq n. \tag{2}$$

By an easy argument based on Polya's disconjugacy condition [**2**], the author showed in [**3**] that (1) is eventually disconjugate if it has a fundamental system $\{y_0, \ldots, y_{n-1}\}$ such that, for $0 \leq m \leq n-1$,

$$y_m^{(r)}(t) = \begin{cases} t^{m-r}(1 + o(1))/(m-r)!, & 0 \leq r \leq m, \\ o(t^{m-r}), & m+1 \leq r \leq n-1. \end{cases} \tag{3}$$

By a theorem of Hartman and Wintner [**1**], (2) implies that (1) has such a fundamental system. In [**3**] the author gave a condition weaker than (2) which implies the same conclusion, but requires the existence of certain auxiliary functions which may not be easy to find. Here we give a condition which is weaker than (2) and imposes only readily verifiable requirements on $p_1, \ldots, p_n$.

2. A preliminary lemma. To state and prove our main results, we need the following lemma.

LEMMA 1. *Suppose $p \in C(0, \infty)$ and $\int^\infty t^{k-1}p(t)\,dt$ converges (perhaps conditionally). Define*

$$I_0(t; p) = p(t) \tag{4}$$

and

$$I_j(t; p) = \int_t^\infty I_{j-1}(s; p)\,ds = \int_t^\infty \frac{(s-t)^{j-1}}{(j-1)!}p(s)\,ds, \qquad 1 \leq j \leq k. \tag{5}$$

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Then the integrals (5) *converge and satisfy the inequalities*

$$|I_j(t;\,p)| \leqslant \frac{2\delta(t)t^{j-k}}{(j-1)!}, \qquad 1 \leqslant j \leqslant k, \tag{6}$$

where

$$\delta(t) = \sup_{T \geqslant t}\left|\int_T^\infty s^{k-1}p(s)\,ds\right| = o(1). \tag{7}$$

The integrals

$$\int^\infty t^{k-j-1}I_j(t;\,p)\,dt, \qquad 0 \leqslant j \leqslant k-1, \tag{8}$$

all converge, and if this convergence is absolute for some j_0 *in* $\{0, 1, \ldots, k-1\}$, *then it is absolute for* $j_0 \leqslant j \leqslant k-1$.

PROOF. Our assumption and Abel's integral test imply that $\int^\infty s^r p(s)\,ds$ converges for $0 \leqslant r \leqslant k-1$; therefore, $I_j(t;\,p)$ converges for $1 \leqslant j \leqslant k$. With $U(t) = \int_t^\infty s^{k-1}p(s)\,ds$,

$$\int_t^\infty (s-t)^{j-1}p(s)\,ds = -\int_t^\infty \left(1-\frac{t}{s}\right)^{j-1} s^{j-k}U'(s)\,ds. \tag{9}$$

If $2 \leqslant j \leqslant k$, integrating the right side by parts yields

$$\int_t^\infty (s-t)^{j-1}p(s)\,ds = \int_t^\infty U(s)\frac{d}{ds}\left[\left(1-\frac{t}{s}\right)^{j-1} s^{j-k}\right] ds.$$

This and routine estimates yield (6) for $2 \leqslant j \leqslant k$, since $|U(s)| \leqslant \delta(t)$ and

$$\left|\frac{d}{ds}\left[\left(1-\frac{t}{s}\right)^{j-1} s^{j-k}\right]\right| \leqslant (k-j)s^{j-k-1} + t^{j-k}\frac{d}{ds}\left(1-\frac{t}{s}\right)^{j-1}$$

if $s \geqslant t$ and $k \geqslant j$. (In fact, we may drop the 2 in (6) if $j = k$.) If $j = 1$, integrating the right side of (9) by parts yields

$$\int_t^\infty p(s)\,ds = t^{-k+1}U(t) - (k-1)\int_t^\infty s^{-k}U(s)\,ds,$$

which implies (6) with $j = 1$.

From (4) and (5), integration by parts yields

$$\int_{t_1}^{t_2} t^{k-j-1}I_j(t;\,p)\,dt = \frac{t^{k-j}}{k-j}I_j(t;\,p)\bigg|_{t_1}^{t_2} + \frac{1}{k-j}\int_{t_1}^{t_2} t^{k-j}I_{j-1}(t;\,p)\,dt$$

if $1 \leqslant j \leqslant k-1$; hence, (6) and the convergence of (8) for $j = 0$ imply that the integrals in (8) all converge, by finite induction. If

$$\int^\infty t^{k-j-1}|I_j(t;\,p)|\,dt < \infty \tag{10}$$

for some $j < k-1$, then

$$\int_t^\infty |I_j(s;\,p)|\,ds = o(t^{-k+j+1}). \tag{11}$$

Now,

$$\int_{t_1}^{t_2} t^{k-j-2}\left(\int_t^\infty |I_j(s;\, p)|ds\right) dt = \frac{t^{k-j-1}}{k-j-1}\int_t^\infty |I_j(s;\, p)|ds\Big|_{t_1}^{t_2} + \frac{1}{k-j-1}\int_{t_1}^{t_2} t^{k-j-1}|I_j(t;\, p)|dt.$$

Therefore, (10) and (11) imply that

$$\int^\infty t^{k-j-2}\left(\int_t^\infty |I_j(s;\, p)|ds\right) dt < \infty,$$

which in turn implies that

$$\int^\infty t^{k-j-2}|I_{j+1}(t;\, p)|dt < \infty,$$

since

$$|I_{j+1}(t;\, p)| \leqslant \int_t^\infty |I_j(s;\, p)|ds.$$

(See (5) with j replaced by $j+1$.) This completes the proof.

To see that Lemma 1 has nontrivial applications, let

$$Q(t) = t^{-k}\exp\left[i|\log t|^{\gamma+1}\right] \qquad (\gamma > 0).$$

The substitution $x = (\log t)^{\gamma+1}$ shows that if F is a polynomial with purely real or purely imaginary nonzero constant term, then the real and imaginary parts of

$$\int^\infty t^{k-1}(\log t)^{-j\gamma}F\left((\log t)^{-\gamma}\right)Q(t)\, dt \tag{12}$$

converge conditionally if $0 \leqslant j\gamma \leqslant 1$, or absolutely if $j\gamma > 1$. Moreover, if $t > 1$ and $0 \leqslant j \leqslant k-1$, then

$$I_j(t;\, Q) = t^j(\log t)^{-j\gamma}F_j\left((\log t)^{-\gamma}\right)Q(t) + O\left(t^{-k+j}(\log t)^{-k\gamma}\right). \tag{13}$$

where F_j is a polynomial of the kind mentioned above. (From (4), (13) holds with $j = 0$. If $0 \leqslant r \leqslant k-2$ and (13) holds with $j = r$, then it can be established for $j = r+1$ by means of the first equality in (5) and repeated integration by parts, integrating the exponential factor at each step. Therefore, (13) holds for $0 \leqslant j \leqslant k-1$, by finite induction.) If $j_0 \geqslant 1$ and $1/j_0 < \gamma < 1/(j_0 - 1)$, then multiplying the "0" term in (13) by t^{k-j-1} produces an absolutely integrable term (since $k\gamma > 1$). Therefore, (13) and the stated convergence properties of (12) imply that the real and imaginary parts of $\int^\infty t^{k-j-1}I_j(t;\, Q)\, dt$ converge conditionally if $0 \leqslant j \leqslant j_0 - 1$, or absolutely if $j_0 \leqslant j \leqslant k-1$. Hence, if

$$p(t) = t^{-k}\cos\left[|\log t|^{\gamma+1}\right] \text{ or } p(t) = t^{-k}\sin\left[|\log t|^{\gamma+1}\right],$$

then $\int^\infty t^{k-j-1}I_j(t;\, p)\, dt$ has these properties.

3. Main results. The following is our main theorem.

THEOREM 1. *Suppose $p_1,\dots,p_n \in C(0,\infty)$. Let the integrals*

$$\int^\infty t^{k-1}p_k(t)\,dt, \qquad 1 \leqslant k \leqslant n,$$

converge, and

$$\int^\infty |I_{k-1}(t;\,p_k)|\,dt < \infty, \qquad 1 \leqslant k \leqslant n. \tag{14}$$

Then (1) *is eventually disconjugate.*

Theorem 1 follows from the next theorem, by the argument given in [**3**].

THEOREM 2. *The assumptions of Theorem* 1 *imply that* (1) *has a fundamental system* $\{y_0,\dots,y_{n-1}\}$ *which satisfies* (3).

PROOF. Let m be a fixed integer, $0 \leqslant m \leqslant n-1$. (Some quantities below depend upon m, but we will not make this explicit in the notation.) For $t_0 > 0$, let $B(t_0)$ be the Banach space of functions y in $C^{(n-1)}[t_0,\infty)$ such that

$$y^{(r)}(t) = O(t^{m-r}), \qquad 0 \leqslant r \leqslant n-1,$$

with norm

$$\|y\| = \sup_{t \geqslant t_0}\left\{\sum_{r=0}^{n-1} t^{r-m}|y^{(r)}(t)|\right\}. \tag{15}$$

Let $My = \sum_{k=1}^n p_k y^{(n-k)}$. We first show that the transformation T defined by

$$(Ty)(t) = 1 + \int_t^\infty \frac{(t-s)^{n-1}}{(n-1)!}(My)(s)\,ds \quad \text{if } m = 0, \tag{16}$$

or by

$$(Ty)(t) = \frac{t^m}{m!} + \int_{t_0}^t \frac{(t-\lambda)^{m-1}}{(m-1)!}\,d\lambda \int_\lambda^\infty \frac{(\lambda-s)^{n-m-1}}{(n-m-1)!}(My)(s)\,ds \tag{17}$$

if $m = 1,\dots,n-1$, is a contraction mapping of $B(t_0)$ into itself if t_0 is sufficiently large. Consider the integral

$$J(t;\,h) = \int_t^\infty s^{n-m-1}(Mh)(s)\,ds, \qquad h \in B(t_0),\ t \geqslant t_0. \tag{18}$$

Using (5) and repeated integration by parts yields

$$\begin{aligned}\int_t^{\bar t} s^{n-m-1}(Mh)(s)\,ds &= \sum_{k=1}^n \int_t^{\bar t} s^{n-m-1}p_k(s)h^{(n-k)}(s)\,ds \\ &= -\sum_{k=2}^n \sum_{j=1}^{k-1} I_j(s;\,p_k)\left[s^{n-m-1}h^{(n-k)}(s)\right]^{(j-1)}\Big|_t^{\bar t} \\ &\quad + \int_t^{\bar t}\left(\sum_{k=1}^n I_{k-1}(s;\,p_k)\left[s^{n-m-1}h^{(n-k)}\right]^{(k-1)}\right)ds.\end{aligned} \tag{19}$$

By using Liebniz' formula for the derivatives of a product, and rearranging terms, we can rewrite the integrand in the last member of (19) as

$$\sum_{j=0}^{n-m-1}[s^{n-m-1}]^{(j)}h^{(n-j-1)}(s)\sum_{k=j+1}^{n}\binom{k-1}{j}I_{k-1}(s;\,p_k).$$

Therefore, (14) implies that this integral converges absolutely at $\bar{t}\to\infty$, because

(20) $$\left|[s^{n-m-1}]^{(j)}h^{(n-j-1)}(s)\right|\leqslant K_j\|h\|,\qquad 0\leqslant j\leqslant n-m-1,$$

where K_j is a universal constant (see (1.5)). Moreover, from (6) with $p=p_k$,

(21) $$|I_j(t;\,p_k)|\leqslant\frac{2\delta_k(t)t^{j-k}}{(j-1)!},\qquad 1\leqslant j\leqslant k,$$

with δ_k as in (7) with $p=p_k$. From (15),

(22) $$\left|[s^{n-m-1}h^{(n-k)}(s)]^{(j-1)}\right|\leqslant B_{jk}\|h\|t^{k-j},\qquad 1\leqslant j\leqslant k-1,$$

where B_{jk} is a universal constant. Therefore, letting $t\to\infty$ in (19) and applying (14), (20), (21) and (22) shows that (18) converges and

(23) $$|J(t;\,h)|\leqslant\sigma(t)\|h\|,\qquad t\geqslant t_0,$$

where

$$\sigma(t)=2\sum_{k=2}^{n}\left(\sum_{j=1}^{k-1}\frac{B_{jk}}{(j-1)!}\right)\delta_k(t)+\sum_{j=0}^{n-m-1}K_j\int_t^{\infty}\left|\sum_{k=j+1}^{n}\binom{k-1}{j}I_{k-1}(s;\,p_k)\right|ds.$$

Since $J(t;\,h)$ converges, Lemma 1 with $p=Mh$ and $k=n-m$ implies that the function $\hat{h}$ defined by

(24) $$\hat{h}(t)=\int_t^{\infty}\frac{(t-s)^{n-1}}{(n-1)!}(Mh)(s)\,ds\quad\text{if } m=0,$$

or by

(25) $$\hat{h}(t)=\int_{t_0}^{t}\frac{(t-\lambda)^{m-1}}{(m-1)!}d\lambda\int_{\lambda}^{\infty}\frac{(\lambda-s)^{n-m-1}}{(n-m-1)!}(Mh)(s)\,ds$$

if $m=1,\ldots,n-1$, is defined for $t\geqslant t_0$. We will now show that $\hat{h}\in B(t_0)$, and estimate $\|\hat{h}\|$. Differentiating (24) or (25) yields

(26) $$\hat{h}^{(r)}(t)=\int_t^{\infty}\frac{(t-s)^{n-r-1}}{(n-r-1)!}(Mh)(s)\,ds,\qquad m\leqslant r\leqslant n-1,$$

so Lemma 1 and (23) imply that

(27) $$|\hat{h}^{(r)}(t)|\leqslant\frac{2\|h\|\sigma(t)t^{m-r}}{(n-r-1)!},\qquad m\leqslant r\leqslant n-1.$$

If $m \geqslant 1$, then we must also estimate $\hat{h}^{(r)}(t)$, $0 \leqslant r \leqslant m-1$. From (25) and (26) (the latter with $r = m$),

$$\hat{h}(t) = \int_{t_0}^{t} \frac{(t-\lambda)^{m-1}}{(m-1)!} \hat{h}^{(m)}(\lambda)\, d\lambda.$$

Differentiating this and invoking (27) with $r = m$ yields

$$|\hat{h}^{(r)}(t)| \leqslant \frac{2\|h\|}{(m-r-1)!\,(n-m-1)!} \int_{t_0}^{t} (t-\lambda)^{m-r-1} \sigma(\lambda)\, d\lambda, \tag{28}$$

$0 \leqslant r \leqslant m-1$. Since σ is nonincreasing, this implies that

$$|\hat{h}^{(r)}(t)| \leqslant \frac{2\|h\|\sigma(t_0) t^{m-r}}{(m-r)!\,(n-m-1)!}, \qquad 0 \leqslant r \leqslant m-1. \tag{29}$$

Now (27) and (29) imply that $\hat{h} \in B(t_0)$. From (16) and (24) or (17) and (25), this implies that $Ty \in B(t_0)$ if $y \in B(t_0)$. Moreover, if y and $\tilde{y}$ are in $B(t_0)$, then (27) and (29) with $h = y - \tilde{y}$ imply that $\|Ty - T\tilde{y}\| \leqslant K\sigma(t_0)\|y - \tilde{y}\|$, where K is a universal constant. Now choose t_0 so that $K\sigma(t_0) < 1$. Then T is a contraction mapping of $B(t_0)$ into itself, and therefore T has a fixed point (function) y_m such that, for $t \geqslant t_0$.

$$y_0(t) = 1 + \int_{t}^{\infty} \frac{(t-s)^{n-1}}{(n-1)!} (My_0)(s)\, ds \quad \text{if } m = 0, \tag{30}$$

(see (16)), or

$$y_m(t) = \frac{t^m}{m!} + \int_{t_0}^{t} \frac{(t-\lambda)^{m-1}}{(m-1)!} d\lambda \int_{\lambda}^{\infty} \frac{(\lambda - s)^{n-m-1}}{(n-m-1)!} (My_m)(s)\, ds \tag{31}$$

if $m = 1, \ldots, n-1$ (see (17)). Clearly, $y_0, \ldots, y_{n-1}$ all satisfy (1) on (t_0, ∞), and they can be extended over $(0, \infty)$ as solutions of (1). If $h = y_m$ and $\hat{h}$ is the integral on the right of (30) or (31) (see (24) and (25)), then (27) implies (3) for $m \leqslant r \leqslant n-1$. If $m \geqslant 1$, then (28) implies that

$$t^{r-m}|h^{(r)}(t)| \leqslant \frac{2\|h\|t^{-1}}{(m-r-1)!\,(n-m-1)!} \int_{t_0}^{t} \sigma(\lambda)\, d\lambda, \qquad 0 \leqslant r \leqslant m-1,$$

which implies (3) for $0 \leqslant r \leqslant m-1$, since the right side here approaches zero as $t \to \infty$. (This is obvious if $\int^{\infty}\sigma(\lambda)\,d\lambda < \infty$; if $\int^{\infty}\sigma(\lambda)\,d\lambda = \infty$, then it follows from l'Hospital's rule, since $\lim_{t\to\infty} \sigma(t) = 0$.) This completes the proof of Theorem 2.

References

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Department of Mathematical Sciences, Drexel University, Philadelphia, Pennsylvania 19104