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REMARKS ON GEODESICS

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ABSTRACT. For a C^1 Riemann metric, we investigate questions concerning "curvature" and "C^1 dependence of geodesics on initial conditions".

1. Statement of results. In what follows, it is assumed that the coefficients of the positive definite Riemann metric

$$ds^2 = g_{ij}(u)du^i du^j, \quad \text{where } g_{ij} = g_{ji}, \tag{1.1}$$

are of class C^1 on an open $u = (u^1, \ldots, u^n)$ neighborhood of $u = 0$. The initial value problem for geodesics of (1.1) is

$$\begin{gathered} u^{i''} + \Gamma^i_{jk}(u)u^{j'}u^{k'} = 0, \qquad i = 1, \ldots, n, \\ u(0) = u_0, \qquad u'(0) = u'_0\ (\neq 0), \end{gathered} \tag{1.2}$$

where the Christoffel symbols $\Gamma^i_{jk} = \Gamma^i_{kj}$ are, of course, continuous. We shall say that (1.1) [or (1.2)] has property (*) at $u = 0$ when the following holds:

(*) There exists a neighborhood U of $u = 0$ such that (1.2) has a unique solution $u = u(s) = u(s, u_0, u'_0)$ and $u(s, u_0, u'_0)$, $u'(s, u_0, u'_0)$ are of class C^1 (as functions of their $2n + 1$ arguments on their domain of definition) for arbitrary $u_0 \in U$ and $u'_0 \neq 0$.

THEOREM (II) of [**4**, p. 278] (cf. also [**5**] or [**6**, Theorem 6.1, p. 104]) states that if $f(s, x)$ is continuous on an open $(s, x) = (s, x^1, \ldots, x^m)$ set E, then a necessary and sufficient condition for

$$x' = f(s, x), \qquad x(f_0) = x_0 \tag{1.3}$$

to have a unique solution $x = \eta(s, s_0, x_0)$, which is of class C^1 as a function of its $m + 2$ arguments, is that, in the vicinity of every point $(s_0, x_0) \in E$, there exists a continuous, nonsingular $m \times m$ matrix $A(s, x)$ such that

$$\omega = A(s, x)[dx - f(s, x)ds] \tag{1.4}$$

has a continuous exterior derivative. (A 1-form ω with continuous coefficients is said to have a continuous exterior derivative if there exists a 2-form, say $d\omega$, with continuous coefficients for which Stokes' formula $\int_{\partial\Sigma} \omega = \iint_\Sigma d\omega$ holds for all admissible 2-surfaces Σ; cf., e.g., [**5**] or [**6**, pp. 101–104].)

It was pointed out in [**4**, pp. 282–284] (cf. [**6**, Exercise 6.2, pp. 106, 563]) that the result just quoted implies that (1.1) has property (*) if (1.1) has a continuous

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Riemann curvature tensor, that is, if the 1-forms

$$\omega^i_{k0} = \Gamma^i_{kj}(u)du^j \quad \text{for } i, k = 1,\dots,n, \tag{1.5}$$

have continuous exterior derivatives. This follows by letting $x = (u, v) = (u, u')$ be a $2n$-vector, identifying (1.3) with (1.2), that is, with the first order system

$$u^{i'} = v^i, \quad v^{i'} = -\Gamma^i_{jk}(u)v^jv^k \qquad \text{for } i = 1,\dots,n, \tag{1.6}$$

and choosing A in (1.4) to be the $2n \times 2n$ matrix

$$A = \begin{pmatrix} I & 0 \\ B & I \end{pmatrix}, \tag{1.7}$$

where $B = B(u, v)$ is the $n \times n$ matrix $(b_{ij}) = (-v^k\Gamma^i_{kj}(u))$ for $i, j = 1,\dots,n$.

In view of the theorem concerning (1.3) quoted above, it is natural to ask if (1.1) has property (*) if and only if it has a continuous Riemann curvature tensor. The answer is in the negative; cf. Proposition 1.2. The considerations below would seem to indicate that there is no reasonable *geometric* condition on (1.1) or (1.2) which is necessary and sufficient for (*). The *analytic* condition provided by [**4**, (II)] is that on a neighborhood of $(s, u, v) = (0, 0, v_0)$ there exist continuous $n \times n$ matrices $(b_{ij}(s, u, v))$, $(c_{ij}(s, u, v))$ such that $\det(c_{ij}) \neq 0$ and

$$\omega_i = b_{ij}\left[du^j - v^jds\right] + c_{ij}\left[dv^j + \Gamma^j_{pq}v^pv^qds\right], \qquad i = 1,\dots,n, \tag{1.8}$$

have continuous exterior derivatives. We first note that the geometric sufficient condition involving (1.5) can be generalized as follows.

THEOREM 1.1. *A sufficient condition for* (1.1) *to have property* (*) *at* $u = 0$ *is that, on an open neighborhood* U *of* $u = 0$, *there exist continuous functions* $\Delta^i_{jk}(u) = -\Delta^i_{kj}(u)$ *for* $i, j, k = 1,\dots,n$ *such that the connection* $\Gamma^i_{jk} + \Delta^i_{jk}$ *has a continuous curvature tensor, that is, the* 1*-forms*

$$\omega^i_k = \left[\Gamma^i_{kj}(u) + \Delta^i_{kj}(u)\right]du^j \quad \text{for } i, k = 1,\dots,n, \tag{1.9}$$

have continuous exterior derivatives on U.

This follows by choosing (1.7) with $B = (b_{ij}(u, v))$, $b_{ij} = -v^k[\Gamma^i_{kj}(u) + \Delta^i_{kj}(u)]$. It also follows from the fact that (*) does not depend on the g_{ij} but rather on the Γ^i_{jk}, and (1.2) is not altered if Γ^i_{jk} is replaced by $\Gamma^i_{jk} + \Delta^i_{jk}$.

The sufficient condition involving (1.9) is more general than the case $\Delta^i_{jk} = 0$, i.e., the condition involving (1.5). In fact, we shall give an example proving the following:

PROPOSITION 1.2. *There exist* C^1 *metrics* (1.1), *with* $n = 2$, *satisfying the conditions of Theorem* 1.1, *hence having property* (*), *but not having a continuous curvature.*

The sufficient condition for (*) given by Theorem 1.1 is not necessary.

THEOREM 1.3. *In Theorem* 1.1, *the requirement that* (1.9) *have a continuous exterior derivative on* U *can be relaxed as follows: there exists a closed set* C *such that* (1.9) *is* S*-Lipschitz continuous on* U *and has a continuous exterior derivative on* $U \setminus C$ *and, for every geodesic arc* Γ *of* (1.1), *the intersection* $\Gamma \cap C$ *has* s*-measure* 0.

This follows from Theorem A in the Appendix which is a generalization of part of (II) in [4] (i.e., [6, Theorem 6.1, p. 104]). It is a corrected version of (III) *in* [7] *which is false*. Theorem 1.3 is more general than Theorem 1.1 in the following sense.

PROPOSITION 1.4. (i) *There exist metrics* (1.1) *of class* C^1, *with* $n = 2$, *satisfying the conditions of Theorem* 1.3 *with* $\Delta^i_{jk} \equiv 0$, *hence having property* (*), *but not satisfying the conditions of Theorem* 1.1 *for any choice of the continuous functions* $\Delta^i_{jk}(u) = -\Delta^i_{kj}(u)$. (ii) *Theorem* 1.3 *may be false even if there is only one geodesic arc* Γ_0 *such that* $\Gamma_0 \cap C$ *has a positive* (*arclength*) *measure*.

2. Proof of Proposition 1.2. Let $u = (u^1, u^2) = (x, y)$. An example (1.1) proving Proposition 1.2 will be chosen in the conformal form

$$ds^2 = e^{z(x,y)}(dx^2 + dy^2), \tag{2.1}$$

where $z(x, y)$ is of class C^1 for small $|x|, |y|$. Note that when $z \in C^2$, the curvature of (2.1) is

$$K = -\tfrac{1}{2}e^{-z}(z_{xx} + z_{yy}). \tag{2.2}$$

The function $z(x, y)$ will be furnished by the following:

On $D = \{(x, y)\colon x^2 + y^2 \leq 1\}$, *there exists a continuous function* $p(x, y)$ *with the property that the "Poisson" equation*

$$2z_{xx} + z_{yy} = p(x, y) \tag{2.3}$$

has a $C^1(D)$-*solution* z *such that* z_{xy} *exists and is continuous, but* z_{xx}, z_{yy} *do not exist at* $(x, y) = (0, 0)$.

This is clear from the criteria in [9] for the existence and continuity of second order partials of logarithmic potentials. (2.3) is understood in the L^2 sense.

Suppose, if possible, that the conformal metric (2.1) has a continuous curvature $K(x, y)$ on some open set U in D. Then (2.2) holds in a (local) L^2 sense on U; cf. [6, Lemma 5.1, p. 102 and Exercise 6.2, p. 106]. Thus (2.2) and (2.3) imply that the L^2 functions z_{xx}, z_{yy} are equal (almost everywhere) to continuous functions on U. But this is not the case.

It remains to show that (2.1) satisfies the conditions of Theorem 1.1 The differential equations for the geodesics of (2.1) are

$$2x'' + z_k x'^2 + 2z_y x'y' - z_x y'^2 = 0, \qquad 2y'' - z_y x'^2 + 2z_x x'y' + z_y y'^2 = 0.$$

Hence, the forms (1.9) are given by

$$2\omega^1_1 = z_x dx + \left(z_y + 2\Delta^1_{12}\right)dy, \quad 2\omega^1_2 = \left(z_y + 2\Delta^1_{21}\right)dx - z_x dy,$$
$$2\omega^2_1 = -z_y dx + \left(z_x + 2\Delta^2_{12}\right)dy, \quad 2\omega^2_2 = \left(z_x + 2\Delta^2_{21}\right)dx + z_y dy,$$

where $\Delta^i_{jj} = 0$. Let $\Delta^1_{12} = -\Delta^1_{21} = z_y/4$ and $\Delta^2_{12} = -\Delta^2_{21} = z_x/2$, so that

$$4\omega^1_1 = 2z_x dx + 3z_y dy, \quad 4\omega^1_2 = z_y dx - 2z_x dy,$$
$$2\omega^2_1 = -z_y dx + 2z_x dy, \quad 4\omega^2_2 = z_x dx + 2z_y dy.$$

These forms have continuous exterior derivatives since $d(z_x dx) = -z_{xy} dxdy$, $d(z_y dy) = z_{xy} dxdy$, and $d(z_y dx - 2z_x dy) = -p(x, y)dxdy$.

3. Proof of Proposition 1.4. Let $u = (u^1, u^2) = (x, y)$ and let (1.1) be the metric

$$ds^2 = dx^2 + f(x)dy^2, \tag{3.1}$$

where $f(x) \equiv f_0(x) = 1 + x + x|x|$ or $f(x) \equiv f_1(x) = 1 + x|x|$, and $2|x| < 1$. Up to the addition of a continuous function of x, the curvature of (3.1) is $f_{xx}/2f = (\operatorname{sgn} x)/f(x)$. Thus (3.1) has a bounded curvature, continuous except on the (nonempty) set $C = \{(x, y): x = 0\}$. The differential equations for the geodesics of (3.1) are

$$2x'' - f_x(x)y'^2 = 0, \qquad (f(x)y')' = 0. \tag{3.2}$$

Hence, on any geodesic, $x'' \geqslant 0$ and either $y' \equiv 0$ or $y' \neq 0$. In the first of these cases, we have $y' \equiv 0$ and $x' \neq 0$, so that the corresponding geodesic meets C in one point.

In the case where $y' \neq 0$ and $f = f_0(x)$ with $f_x(x) = 1 + 2|x| > 0$, we have $x'' > 0$, so that the geodesic meets C in at most two points. Hence, (3.1) with $f = f_0$ satisfies the conditions of Theorem 1.3 with $\Delta^i_{jk} = 0$.

In the case $f = f_1(x)$ with $f_x(0) = 0$, the line C is a geodesic and all other geodesics meet C in at most two points. Hence, except for the geodesic $\Gamma_0 = C$, the conditions of Theorem 1.3 are satisfied. It was shown, however, in [**8**, pp. 329–330], that for (3.1) with $f = f_1$, the geodesics are uniquely determined by initial conditions but are not dependent on them in a C^1 manner. This example proves Proposition 1.4(ii).

In order to complete the proof of (i), it must be shown that (3.1) with $f = f_0$ does not satisfy the conditions of Theorem 1.1. If it does, then there exists a continuous function $g(x, y) = 2\Delta^1_{21}$ such that

$$2\omega^1_2 = g dx - f_x dy = g dx - (1 + 2|x|)dy$$

has a continuous exterior derivative, say, $-p(x, y)dxdy$. It follows that, for $x \neq 0$, g has a continuous partial derivative g_y and

$$g_y = p(x, y) - f_{xx}(x) = p(x, y) - 2 \operatorname{sgn} x \quad \text{for } x \neq 0;$$

cf. [**6**. Exercise 5.2, pp. 104, 562]. Hence, for $x \neq 0$,

$$g(x, y) = -2y \operatorname{sgn} x + \int_0^y p(x, t)\, dt + g(x, 0).$$

But a function of this form cannot be continuous for small $|x|$, $|y|$.

Appendix. Theorem (II) of [**4**] (i.e., [**6**, Theorem 6.1, p. 104]) gives necessary and sufficient conditions in order that

$$x' = f(s, x), \qquad x(s_0) = x_0, \tag{1}$$

has a unique solution $x = \eta(s, s_0, x_0)$ which is of class C^1 as function of all its arguments. Here we give a "generalization" of the sufficiency part which implies Theorem 1.3 above.

THEOREM A. *Let $f(s, x)$ be continuous on an open $(s, x) = (s, x^1, \ldots, x^m)$ set U with the property that there exists a closed set C and a continuous nonsingular $m \times m$*

matrix $A(s, x)$ on U such that: (i) *the* 1-*form*

$$\omega = A(s, x)[dx - f(x, x)dx] \tag{2}$$

is Lipschitz continuous on U (or is merely S-Lipschitz continuous on U; cf. [**6**, *pp.* 107–109]) *and has a continuous exterior derivative on $U \setminus C$; and* (ii) *every solution arc* $(s, x(s))$ *of* (1) *meets C in a set of s-measure* 0. *Then the initial value problem* (1) *has a unique solution $x = \eta(s, s_0, x_0)$ which is of class C^1 on its $(m + 2)$-dimensional domain of existence.*

The simplest proof of Theorem A is obtained by modifying the arguments of [**6**, p. 113] in the proof of Theorem 6.1. The considerations below make clear what modifications are necessary. In order to avoid repeating the entire proof, we give a slightly different argument using Theorem 6.1 and Corollary 6.1 of [**6**, pp. 104–105].

PROOF OF THEOREM A. According to [**5**] (cf. [**6**, Theorem 8.1, p. 109]), the fact that (2) is S-Lipschitz continuous implies (1) has a unique solution $x = \eta(s, s_0, x_0)$ which is locally uniformly Lipschitz continuous on its domain of existence, say Ω.

The conditions on (2) imply there exists a continuous, bounded $m \times m$ matrix function $F(s, x)$ on $U \setminus C$ with the property that if $(s_0, x_0) \notin C$, then $y = \partial\eta/2x_0^k$, $k = 1, \ldots, m$, exists on any s-interval containing s_0 along which $(s, x) = (s, \eta(s, s_0, x_0)) \notin C$ and is a solution of

$$[A(s, \eta)y]' = F(s, \eta)y, \qquad y(s_0) = e_k, \tag{3}$$

where $e_k = (e_{k1}, \ldots, e_{km})$, $e_{kj} = 0$ for $k \neq j$ and $e_{kk} = 1$, and $\eta = \eta(s, s_0, x_0)$; cf. [**4**, **5**] or [**6**, Corollary 6.1, p. 105].

The matrix function $A(s, \eta)$ is continuous for $(s, s_0, x_0) \in \Omega$ and is nonsingular. Also $F(s, \eta)$ is a bounded continuous function of (s, s_0, x_0) on its domain of definition. For fixed (s_0, x_0), the set of s-values for which $F(s, \eta)$ is not defined is of measure 0. It follows that, whether or not $(s_0, x_0) \in C$, there exists a unique solution $y = y(s) = y(s, s_0, x_0)$ of (3) (in the sense that y is an absolutely continuous function of s satisfying (4) almost everywhere) which is a continuous function of (s, s_0, x_0) defined on the domain Ω of existence of $\eta(s, s_0, x_0)$. This can be verified by a modification of standard arguments involving linear differential equations; cf., e.g., [**6**, proof of Corollary 4.1, p. 55].

Since $\eta(s, s_0, x_0)$ is locally uniformly Lipschitz continuous, the partial derivative $\partial\eta/\partial x_0^k$ exists except on an (s, s_0, x_0)-set of $(m + 2)$-dimensional measure 0. This partial derivative $\partial\eta/\partial x_0^k$ has an extension to the continuous function $y(s, s_0, x_0)$ on Ω. Hence, $\partial\eta/\partial x_0^k$ exists and is continuous on Ω.

In a standard way, it follows that $\partial\eta/\partial s_0$ exists and is the continuous function $\partial\eta/\partial s_0 = -\Sigma(\partial\eta/\partial x_0^k)f^k(s_0, x_0)$ on Ω; cf. [**6**, proof of (3.4), pp. 96, 99]. This completes the proof of Theorem A.

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