SOME TILINGS OF THE PLANE
WHOSE SINGULAR POINTS FORM A PERFECT SET

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Abstract. Let $\mathcal{S}$ be a tiling of the plane such that for every tile $T$ of $\mathcal{S}$ there correspond a tile $T'$ of $\mathcal{S}$ (not necessarily unique) and an integer $k(T, T')$ (depending on $T$ and $T'$), $2 < k$, such that $T$ meets $T'$ in $k(T, T')$ connected components. Then the set of singular points of $\mathcal{S}$ is a nowhere dense, perfect set.

1. Introduction. We begin with some preliminary definitions. The family $\mathcal{S}$ is a tiling for the plane if and only if $\mathcal{S}$ is a collection of closed topological disks having pairwise disjoint interiors for which $\bigcup \{T: T \text{ in } \mathcal{S}\} = \mathbb{R}^2$. Point $p$ in $\mathbb{R}^2$ is a singular point of $\mathcal{S}$ if and only if every neighborhood of $p$ meets infinitely many tiles of $\mathcal{S}$, and a tiling having no singular point is said to be locally finite. The reader is referred to Grünbaum and Shepard [1] for a thorough treatment of these topics.

In [2], Valette examined tilings $\mathcal{S}$ of the plane having the property that for every tile $T$ of $\mathcal{S}$ there correspond a tile $T'$ of $\mathcal{S}$ (not necessarily unique) and an integer $k(T, T')$ (depending on $T$ and $T'$), $2 < k$, such that $T$ meets $T'$ in $k(T, T')$ connected components. While examples reveal that such tilings exist, Valette proved that no such tiling can be locally finite. Here we use his results to obtain the following theorem: For tiling $\mathcal{S}$ having the property above, the set of singular points of $\mathcal{S}$ is a perfect set. That is, the set of singular points of $\mathcal{S}$ is closed and has no isolated points. Furthermore, the set is nowhere dense in the plane.

Throughout the paper, $\text{bdry } S$ will be used to denote the boundary for set $S$.

2. The results.

Theorem 1. Let $\mathcal{S}$ be a tiling of the plane such that for every tile $T$ of $\mathcal{S}$ there correspond a tile $T'$ of $\mathcal{S}$ (not necessarily unique) and an integer $k(T, T')$ (depending on $T$ and $T'$), $2 < k$, such that $T$ meets $T'$ in $k(T, T')$ connected components. Then the set of singular points of $\mathcal{S}$ is a nowhere dense, perfect set.

Proof. The following terminology will be useful. If $T_1$ and $T_2$ are two associated tiles in $\mathcal{S}$ (that is, tiles which satisfy our hypothesis), let $\{D_{12}^i\}$ be the family of closures of the bounded components of complement $(T_1 \cup T_2)$, $1 \leq i \leq k(T_1, T_2) - 1$, and let $Y_{12}$ be the closure of the unbounded component. Refer to the collection $\{D_{12}^i\}$ as a bounded component family (of $T_1$ and $T_2$).

From Valette's work, it follows that each $D_{12}^i$ is a closed topological disk whose boundary consists of two arcs, one in $T_1$ and the other in $T_2$, with only their...
endpoints in common. Call such points junction points of set bdry $D_{12}$. If $D_{12}$ and $D_{22}$ intersect, their intersection is a single point which is a junction point of each boundary. No three members of the collection \{\(D_{i2}, Y_{i2}\)\} have a point in common. Thus if $D_{34}$ and $D_{44}$ are subsets of $D_{12}$, they cannot intersect at a junction point of bdry $D_{12}$, since then $D_{34}, D_{44}$ and $Y_{34}$ would have a point in common. We will digress to establish the following preliminary result.

**Lemma 1.** Let $D$ be a member of any bounded component family. Then $D$ contains infinitely many singular points of \(\mathcal{T}\).

**Proof of Lemma 1.** To reach a contradiction, assume $D$ contains only finitely many singular points of \(\mathcal{T}\). Since \(\mathcal{T}\) is a tiling of the plane, disk $D$ contains bounded component families, and we let \(\{D_{i2}\}\) be a minimal bounded component family contained in $D$. (That is, \(\{D_{i2}\}\) is a bounded component family in $D$ containing the fewest singular points.) Each $D_{i2}$ contains a bounded component family which in turn must be minimal. Adapting a proof of Valette [2, Proposition 1, V], it is not hard to see that each $D_{i2}$ must contain a singular point. Moreover, by our choice of \(\{D_{i2}\}\), each $D_{i2}$ must contain every singular point in their union, so there can be only one such point. It follows that all the $D_{i2}$ must have a unique singular point $x$ in common, and $x$ is a junction point of each set bdry $D_{i2}$. Moreover, by previous comments, $i = 1, 2$. Now $D_{12}$ contains a minimal bounded component family \(\{D_{34}, D_{44}\}\), and the sets $D_{34}$ must intersect in $x$ since $x$ is the only singular point available in $D_{12}$. But this contradicts the observation above that two such sets cannot intersect at a junction point of bdry $D_{12}$. Our assumption is false, and Lemma 1 is established.

We are ready to complete the proof of Theorem 1. We must show that the set of singular points of \(\mathcal{T}\) is perfect. Since the set of singular points of any tiling is closed, it suffices to show that there are no isolated singular points.

Suppose on the contrary that $p$ is an isolated singular point of \(\mathcal{T}\), and let $(M, p)$ be a closed circular disk at $p$ whose boundary $C$ is free of any other singular points. Now $(M, p)$ intersects an infinite number of tiles in \(\mathcal{T}\), and since at most a finite number of these can intersect $C$, there must be an infinite subcollection contained in $(M, p)$. Each member of this subcollection gives rise to a bounded component family, and if any corresponding $D_{ij}$ were completely contained in $(M, p)$, then by the Lemma there would be an infinite number of singular points in $(M, p)$, impossible.

Thus each corresponding $D_{ij}$ must meet $C$, and since $C$ meets at most finitely many tiles, there must be a tile $T$ which serves as the associated tile for an infinite number of tiles in $(M, p)$. Thus bdry $T \cap (M, p)$ contains an infinite set of pairwise different points, one from each of this infinite collection of tiles. Hence bdry $T \cap (M, p)$ contains a singular point. This point must be $p$.

Now note that bdry $T \cap (M, p)$ has an infinite number of components. The same is true of bdry $T \cap (N, p)$ where $(N, p) \subseteq (M, p)$. This means bdry $T$ cannot be locally connected at $p$, which is impossible. Our supposition is false, and \(\mathcal{T}\) cannot have an isolated singular point.
Finally, it is not hard to show that the set of singular points of any tiling is nowhere dense in $\mathbb{R}^2$. This finishes the proof of Theorem 1.

We remark that the number 2 in Valette's result and the number 2 in Theorem 1 above are best by [2, Figure 5]. However, if the components mentioned in the hypothesis are required to be arcs, then one of these bounds may be lowered, and minor modifications in Valette's proof yield the following analogue.

**Corollary to Valette's Proposition 1.** If $\mathcal{T}$ is a tiling of the plane such that every tile meets some other tile in a finite number of components, at least 2 of which are arcs, then $\mathcal{T}$ is not locally finite.

It is interesting to observe that such analogues of Lemma 1 and Theorem 1 fail, as Example 1 illustrates.

**Example 1.** Let $C_1(r)$ and $C_2(r)$ be the closed half-circles of radius $r$ centered at the origin and lying in the upper and lower half-planes, respectively. Let $L$ be the $x$-axis, and let $S = \bigcup \{ C_1(n) \cup C_2(n) \cup C_1(1/n) : n \geq 1 \} \cup L$. Define tiling $\mathcal{T}$ to be the collection of closures of components of $\mathbb{R}^2 \sim S$. Then every tile of $\mathcal{T}$ meets another tile of $\mathcal{T}$ in 2 arcs, yet $\mathcal{T}$ has only one singular point, the origin.

We close with an interesting open problem suggested by the referee: Characterize the perfect sets which occur as the set of singular points of tilings of the type considered here.

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**References**


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