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THE TAIL σ-FIELD OF A FINITELY ADDITIVE MARKOV CHAIN STARTING FROM A RECURRENT STATE

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ABSTRACT. For a Markov chain with an arbitrary nonempty state space, with stationary finitely additive transition probabilities and with initial distribution concentrated on a recurrent state, it is shown that the probability of every tail set is either zero or one. This generalizes and in particular gives an alternative proof of the result due to Blackwell and Freedman [1] in case the state space is countable and all transition probabilities are countably additive.

1. Introduction. Let I be an arbitrary nonempty set and I^* be the set of all finite sequences of elements of I, including the empty one. Let $\sigma = \{\sigma(p)\colon p \in I^*\}$ be a family of finitely additive probabilities defined on all subsets of I. In the terminology of Dubins and Savage [3], the family σ is called a strategy. Following Dubins and Savage [2 and 3] and Purves and Sudderth [7], each strategy σ determines a finitely additive probability on the σ-field $\mathfrak{B}$ of subsets of $H = X \times X \times \cdots$ which is generated by the open sets of H when H is equipped with the product of discrete topologies. This probability on $\mathfrak{B}$ is also denoted by σ. In case all probabilities are countably additive, σ coincides with the probability measure obtained on the product σ-field using the Ionescu-Tulcea theorem [5].

To study Markov chains with stationary transitions, we consider the case when, for each nonempty $p \in I^*$, $\sigma(p) = \sigma(i)$ where i is the last coordinate of p. Such a σ will be called a Markov strategy with stationary transitions. For $p \in I^*$ and $A \in \mathfrak{B}$, let Ap denote $\{h \in H\colon ph \in A\}$ where ph is the element of H whose terms consist of the terms of p followed by the terms of h. Also for a strategy σ and $p \in I^*$, the conditional strategy given p, denoted by $\sigma[p]$, is defined by $\sigma[p](q) = \sigma(pq)$, $q \in I^*$, where pq is the element of I^* whose terms consist of the terms of p followed by the terms of q. We also use the suggestive notation $\sigma(A \mid p)$ for $\sigma[p](Ap)$. For a Markov strategy with stationary transitions σ, call an element $i \in I$ a recurrent state if $\sigma[i]\{h\colon \text{some coordinate of } h \text{ is } i\} = 1$.

A set $A \in \mathfrak{B}$ is called a tail set if $Ap = Aq$ whenever p, q are elements of I^* of the same length. (It is well known that the tail sets form a σ-field called the tail σ-field.) The following is the main result of this paper.

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THEOREM. *Let σ be a Markov strategy with stationary transitions and i be a recurrent state. Then for every tail set A, $\sigma[i](A) = 0$ or 1.*

The special case when I is a countable set and $\sigma(j)$ is countably additive for each $j \in I$ is proved by Blackwell and Freedman [**1**]. Their proof crucially makes use of the Hewitt-Savage zero-one law which is known to be false in the finitely additive case [**8**]. This paper thus provides an alternative proof for their result.

2. Preliminaries. We shall need two important results from the theory of finitely additive probability already known. For $h \in H$ and n a positive integer, let $p_n(h)$ denote the sequence of first n coordinates of h.

Levy zero-one law. Let σ be a strategy and $B \in \mathcal{B}$. Then

$$\sigma\{h \in H: \sigma(B \mid p_n(h)) \to 1_B(h)\} = 1.$$

This result is due to Purves and Sudderth and a proof appears in [**6** and **8**].

To state the other result, let σ be a Markov strategy with stationary transitions and let i be a recurrent state. For each positive integer k, let t_k be defined by

$$t_k(h) = \min\{n: \text{there are } k \text{ occurrences of } i \text{ in } p_n(h)\}.$$

(If the set within braces is empty, t_k is defined as ∞.) Let F be the set of all nonempty finite sequences of elements of I whose last coordinate is i and none of the other coordinates is i. Let $\Omega = F \times F \times \cdots$. Equip Ω with the product of discrete topologies and let $\mathcal{F}$ be the σ-field generated by the open sets of Ω. On $G_i = \{h$: infinitely many coordinates of h are $i\}$, we define a sequence of F-valued functions $\{\beta_n\}$ called the block variables as follows:

$$\beta_1(h) = p_{t_1(h)}(h) \quad \text{and} \quad \beta_{n+1}(h) = \left(h_{t_n(h)+1}, \ldots, h_{t_{n+1}(h)}\right),$$

where h_k denotes the kth coordinate of h, $h \in G_i$ and $n \geqslant 1$. Let Φ be the mapping on G_i into Ω defined by $\Phi(h) = (\beta_1(h), \beta_2(h), \ldots)$. Let γ be the finitely additive probability on F defined by $\gamma(D) = \sigma[i](\beta_1^{-1}(D))$, $D \subseteq F$. γ is a probability because it is known that [**9**, p. 253] i is recurrent if and only if $\sigma[i](G_i) = 1$. Let π be the strategy on F defined by $\pi(q) = \gamma$ for every finite sequence q of elements of F. As before let π also be the finitely additive probability induced by the strategy on $\mathcal{F}$.

BLOCK THEOREM. *For every $B \in \mathcal{F}$,*

$$\pi(B) = \sigma[i](\Phi^{-1}(B)).$$

The proof of the above result can be found in [**9**].

3. Proof. We first prove the following lemma.

LEMMA. *Let σ be a Markov strategy with stationary transitions. Let A be a tail set. Then there exists a sequence $\{E_n\}$ of subsets of I such that*

$$\sigma(A) = \sigma\{h = (h_1, h_2, \ldots): h_n \in E_n \text{ for infinitely many } n\}.$$

PROOF OF LEMMA. For the tail set A, let A_n denote Ap where p is an element of I^* of length n (since A is a tail set Ap depends only on n). Let $E_n = \{j \in I: \sigma[j](A_n) > \frac{1}{2}\}$. Applying the Levy zero-one law to σ and A, if C denotes the set of h for which $\sigma(A \mid p(h)) \to 1_A$, it is easily checked that

$$A \cap C = \{h: h_n \in E_n \text{ for infinitely many } n\} \cap C.$$

This proves the lemma.

We shall call a strategy σ an independent strategy if $\sigma(p) = \gamma_n$ for all sequences p of length n, where $\{\gamma_n\}$ is a sequence of finitely additive probabilities defined on all subsets of I. Clearly for any p of length n, $\sigma[p]$, the conditional strategy σ given p, depends on p only through n.

Kolmogorov zero-one law. Let σ be an independent strategy and let A be a tail set. Then $\sigma(A)$ is either zero or one.

PROOF. For each n, define

$$\begin{aligned} E_n &= H \text{ if } \sigma(A \mid p) > \tfrac{1}{2} \quad \text{for some (hence all) } p \text{ of length } n, \\ &= \phi \quad \text{otherwise.} \end{aligned}$$

It is easily checked using the Levy zero-one law as in the previous lemma that

$$\sigma(A) = \sigma\{h: h_n \in E_n \text{ for infinitely many } n\}.$$

Consequently

$$\begin{aligned} \sigma(A) &= 1, \text{ if } E_n = H \quad \text{for infinitely many } n, \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

REMARK. The Kolmogorov zero-one law, as stated above, is due to Purves and Sudderth and a proof appears in [**6** and **8**]. The proof given above is different from theirs and illustrates the use of the technique used in the proof of the lemma.

PROOF OF THE THEOREM. For the tail set A and the Markov strategy σ, obtain using the lemma, a sequence $\{E_n\}$ of subsets of I such that

$$\sigma[i](A) = \sigma[i]\{h: h_n \in E_n \text{ for infinitely many } n\}.$$

For the recurrent state i, as is already stated in §2, it is known that $\sigma[i](G_i) = 1$ where $G_i = \{h: \text{infinitely many coordinates of } h \text{ are } i\}$. Let F, Ω, $\mathfrak{F}$, $\{t_n\}$, $\{\beta_n\}$, γ and π be defined as in §2. Define a sequence $\{F_n\}$ of subsets of F as follows. For $n \geqslant 1$, let

$$F_n = \{\beta_n(h): h_k \in E_k \text{ for at least one } k \text{ satisfying } t_n(h) + 1 \leqslant k \leqslant t_{n+1}(h)\}$$

where h_k is the kth coordinate of h, $h \in G_i$.

Let $B = \{\omega \in \Omega: \omega_n \in F_n \text{ for infinitely many } n\}$, where ω_n is the nth coordinate of ω.

It is straightforward to check that

$$\Phi^{-1}(B) = \{h: h_n \in E_n \text{ for infinitely many } n\} \cap G_i.$$

Consequently, by the Block Theorem,

$$\begin{aligned} \pi(B) &= \sigma[i](\{h: h_n \in E_n \text{ for infinitely many } n\} \cap G_i) \\ &= \sigma[i](A). \end{aligned}$$

Clearly, π is an independent strategy (in fact i.i.d!) and B is a tail set in Ω. Therefore by the Kolmogorov zero-one law, $\pi(B) = 0$ or 1. That completes the proof of the theorem.

4. Some remarks. Call a set $A \in \mathcal{B}$ an exchangeable set in case $h = (h_1, h_2, \ldots, h_n, \ldots) \in A$ iff $h' = (h_{\pi(1)}, h_{\pi(2)}, \ldots, h_{\pi(n)}, \ldots) \in A$ for every permutation π of the positive integers which leaves all but finitely many positive integers unchanged. The proof of Blackwell and Freedman in the countably additive case for a countable state space in fact yields the following stronger result.

THEOREM. *For every exchangeable set $A \in \mathcal{B}$, if i is a recurrent state and σ is a (countably additive) Markov strategy with stationary transitions, $\sigma[i](A)$ is either zero or one.*

This result is false in the general case (and hence our main theorem requires a different proof from theirs) as the following example shows.

EXAMPLE. Let I be the set of all positive integers and σ be the strategy with transitions defined as follows: $\sigma(1) = \gamma$, where γ is a finitely additive probability which is zero for all finite sets and γ {all even numbers} $= \frac{1}{2}$; for $n \geqslant 2$, $\sigma(n) = \delta_1$, the point mass at 1. Under σ, 1 is a recurrent state. Let

$$A = \{h\colon \min(h_n\colon h_n \geqslant 2, n \geqslant 1) \text{ is even}\}.$$

Note that A is an exchangeable set. Using Lemma 7.1 of **[7]**, it can be shown that

$$\sigma[1]\{h\colon h_{2n} = 1 \text{ for } n \geqslant 1 \text{ and } h_1 < h_3 < h_5 < \cdots\} = 1.$$

Consequently,

$$\sigma[1](A) = \sigma[1]\{h\colon h_1 \text{ is even}\} = \gamma(\text{even numbers}) = \tfrac{1}{2}.$$

This example uses the same idea of Purves and Sudderth **[8]** for a counterexample to the Hewitt-Savage zero-one law.

Purves and Sudderth have shown in **[6]** that if σ is an independent strategy, then σ is countably additive when restricted to the tail σ-field. It easily follows from this result and the Block Theorem that if σ is a Markov strategy and $\mathcal{C}$ is the class of all tail sets A in $\mathcal{B}$ for which $\Phi(A \cap G_i)$ is a tail set in $\mathcal{F}$, then $\sigma[i]$ is countably additive when restricted to $\mathcal{C}$. It is worth noting that $\mathcal{C}$ is not in general equal to the tail σ-field in $\mathcal{B}$. However it always includes the invariant σ-field, namely all those sets A in $\mathcal{B}$ for which $h \in A$ iff $ph \in A$ for every $p \in I^*$. It is not known in general whether $\sigma[i]$ is countably additive when restricted to the whole of the tail σ-field.

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