EXTENDING FAMILIES OF DISJOINT ZERO SETS

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ABSTRACT. The \( z(X) \) of a space \( X \) is defined as

\[
\text{\( z(X) = \sup \{ |Z| : Z \subseteq Z(X) \} \)}
\]

where \( Z(X) \) is the family of zero sets of \( X \). It is proved using \( CH \) that a Tychonoff space \( S \) is \( TC^* \)-embedded in every Tychonoff space it is \( C \)-embedded in iff \( z(S) \leq c \). A space \( S \) is defined to be \( TC^* \)-embedded in a space \( X \) if any disjoint family of zero sets of \( S \) can be extended to a family of disjoint zero sets of \( X \). Similar theorems are proved for \( C^* \)-embedding when \( S \) is a \( P \)-space or the zero sets have the Isiwata property.

1. Introduction. If a set \( S \) is dense and \( C \)-embedded in a Tychonoff space \( X \) the closure of a zero set of \( S \) in \( X \) is a zero set of \( X \) [6]. Hence any family of disjoint zero sets of \( S \) may be extended to a family of disjoint zero sets of \( X \). A set \( A \) of \( S \) is said to be extended to a set \( E(A) \) of \( X \) if \( E(A) \cap S = A \).

In this paper, we investigate the disjoint extension of families of disjoint zero sets of a set to a space, some families with restricted cardinals and some with restrictions to regular zero sets. Unless noted, definitions and terminology will be found in [6].

THEOREM 1. A space \( S \) is \( C^* \)-embedded in a space \( X \) iff any denumerable family of disjoint zero sets may be extended disjointly to a family of zero sets of \( X \).

PROOF. Let \( \{Z_n\} \) be a disjoint denumerable family of zero sets of \( X \). Let \( H_{nk} \) be a zero set extension of \( Z_n \) to \( X \) disjoint from \( H_{kn}, k \neq n \), guaranteed by the \( C^* \)-embedding of \( S \) in \( X \). Then the family \( \{H_n : H_n = \bigcap \{H_{nk} : k \neq n\} \} \) is the desired family of disjoint extensions of \( \{Z_n\} \).

DEFINITION 1. A set \( S \) is \( TC^* \)-embedded (\( T^* \)-embedded) in a space \( X \) if any disjoint family of zero (cozero) sets of \( S \) can be extended to a disjoint family of zero (cozero) sets of \( X \).

In [1], \( T^* \)-embeddings were studied.

It is clear that any \( C^* \)-embedded zero set is \( TC^* \)-embedded and if every subset of a space is \( TC^* \)-embedded, then the space is perfectly normal and extremally disconnected.

2. C-embeddings and \( TC^* \)-embeddings. We can improve Theorem 1 with respect to \( TC^* \)-embeddings when the embedded set is \( C \)-embedded and dense as mentioned earlier. More generally we have the following theorem.

THEOREM 2. Let \( S \) be \( C \)-embedded in a Tychonoff space \( X \). Then \( \omega_1 \) disjoint zero sets in \( S \) may be extended to disjoint zero sets of \( X \).

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Proof. We order the $\omega_1$ zero sets \{\(Z_\alpha\)\} by putting them into one-to-one correspondence with the countable ordinals. Let \(E(Z_\alpha)\) be a zero set extension of \(Z_\alpha\) for \(\alpha < \omega_1\). By means of Lemma 2, we construct a zero set \(Z_\beta^\alpha\) of \(X\) such that \(Z_\alpha \subset Z_\beta^\alpha\) and \(Z_\beta^\alpha \cap E(Z_\beta) = \emptyset\) for each \(\beta < \alpha\). The family \{\(H_\alpha\)\}, \(H_\alpha = \bigcap_\beta \{Z_\beta^\beta : \beta < \alpha\} \cap E(Z_\alpha)\) is a family of disjoint zero set extensions of \{\(Z_\alpha\)\}.

Lemma 2. A \(z\)-embedded set \(S\) is \(C\)-embedded in a space \(X\) iff for \(H\) a zero set of \(S\) and \(Z\) a zero set of \(X\), \(H \cap Z = \emptyset\), \(H\) and \(Z\) are completely separated in \(X\).

Proof. Sufficiency. Since \(S\) is \(z\)-embedded and completely separated from any disjoint zero set of \(X\), \(S\) is \(C\)-embedded in \(X\) [3].

Necessity. Let \(f \in C(X)\) and \(h \in C(S)\) such that \(Z = f^{-1}(0)\) and \(H = h^{-1}(0)\). Then \(|f| + |h|\) is continuous on \(S\) and can be extended to a positive continuous function \(g\) on \(X\). Set \(h' = g - |f|\). Then \(h' = |h|\) on \(S\). Since \(g > 0\), \(h' > 0\) on \(Z\). So \(h'^{-1}(0)\) and \(Z\) are disjoint zero sets of \(X\) containing \(H\) and \(Z\) respectively.

Remark 1. It is clear from the above proof that \(C\)-embedding of a set \(S \subset X\) is equivalent to \(C^*\)-embedding and every positive \(f \in C^*(X)\) having a positive continuous extension. This is probably known.

Definition 2. (\(z\)-cellularity) \(z(X) = \sup\{|Z| : Z \subset Z(X), Z\) a disjoint family\} where \(Z(X)\) is the family of all zero sets of \(X\).

Corollary 2. If \(X\) is almost compact [6] with \(z(X) = \omega_1\), then \(X\) is \(TC^*\)-embedded in every Tychonoff space \(X\) is embedded in.

The necessity of the above conditions follows from the next theorem.

Theorem 3 (CH). A Tychonoff space \(X\) is \(TC^*\)-embedded in every Tychonoff space it is \(C\)-embedded (\(C^*\)-embedded) in iff (only if) \(z(X) \leq c\).

Proof. Sufficiency of the \(C\)-embedding result follows from Theorem 2. Necessity is obtained by embedding \(\nu X\) in the product of real lines, in which \(X\) will be \(C\)-embedded in and by Engelking's [4, p. 295] result that \(z(X) \leq c\) for the product of regular separable spaces. Thus if \(z(X) > c\), there is a Tychonoff space in which \(X\) is \(C\)-embedded in but not \(TC^*\)-embedded in. The \(C^*\)-embedding result follows.

Remark 2. We cannot replace \(C\)-embedding by \(C^*\)-embedding in the sufficiency argument as Ralph Fox has shown that \(R\) is not \(TC^*\)-embedded in \(\beta R\).

Corollary 3 (CH). A Tychonoff space \(X\) is \(TC^*\)-embedded in every Tychonoff space it is embedded in iff \(X\) is almost compact and \(z(X) \leq c\).

2. \(C^*\)-embeddings and \(TC^*\)-embeddings.

Theorem 4. If a family of \(\omega_1\) disjoint zero sets of a Tychonoff space \(S\) may be extended disjointly to \(\beta S\), then this family of zero sets of \(S\) may be disjointly extended to any Tychonoff space \(S\) is \(C^*\)-embedded in.

Proof. If \(S\) is \(C^*\)-embedded in \(X\), \(\beta S \subset \beta X\) and \(\beta S\) is \(C\)-embedded in \(\beta X\). By Theorem 2 \(\omega_1\) disjoint zero sets of \(\beta S\) may be extended to \(\beta X\). The result follows.

Corollary 4 (CH). Let \(S\) be Tychonoff and \(TC^*\)-embedded in \(\beta S\). Then \(S\) is \(TC^*\)-embedded in every space it is \(C^*\)-embedded in iff \(z(S) \leq c\).
It is not unusual to have a set $S$ $TC^*$-embedded in $\beta S$ even when there are families of disjoint zero sets of $S$ of large cardinality. This is always true when $S$ is pseudocompact or discrete. In these cases, by Theorem 4, any family of $\omega_1$ disjoint zero sets may be extended disjointly to a family of zero sets of any Tychonoff space in which $S$ is $C^*$-embedded in. The next two theorems are modifications of these results.

**Theorem 5.** Let $\{Z_\alpha\}$ be a family of disjoint zero sets of cardinality $\omega_1$ of a set $S$ such that for $f \in C^*(S)$ such that $Z(f) \cap Z_\alpha = \emptyset$, $|f(z)| \geq \epsilon > 0$ for $z \in Z_\alpha$. Then $\{Z_\alpha\}$ may be extended disjointly to any space in which $S$ is $C^*$-embedded.

**Proof.** Let $E(Z_\alpha)$ be a zero set extension of $Z_\alpha$ to $X$. Thus as in Lemma 2, $E(Z_\alpha)$ is completely separated in $X$ from $Z_\beta$ for $\beta \neq \alpha$. The proof is completed as in Theorem 2.

The property referred to at the end of the first sentence of the statement of the theorem was discovered by Isiwata [7] to be satisfied by all zero sets of a pseudocompact space. We will refer to the above property as the Isiwata property. By modifying the proof of Theorem 5, one may prove the following result.

**Theorem 5A.** Let $S \subset X$, $X$ Tychonoff, have at most $\omega_1$ disjoint zero sets and let every disjoint family of zero sets, none of which has the Isiwata property be denumerable. Then $S$ is $TC^*$-embedded in $X$ if $S$ is $C^*$-embedded in $X$.

**Definition 3 (compare [2]).** A weak 0$\Sigma$-space is a space where the closure of a cozero set is a zero set.

We will call a zero set that is also a regular closed set a regular zero set.

**Theorem 6.** Let $\beta S$ be weak 0$\Sigma$. If $\{Z_\alpha\}$ is a disjoint family of regular zero sets (of cardinality $\omega_1$) then $\{Z_\alpha\}$ may be extended disjointly to a family of zero sets of $\beta S$ (of $X$ Tychonoff where $S$ is $C^*$-embedded in $X$).

**Proof.** Since $\beta S$ is weak 0$\Sigma$, $S$ is weak 0$\Sigma$. So the interior of $Z_\alpha$ is a cozero set. From [1], $\{\text{int} Z_\alpha\}$ has a family of disjoint cozero extensions $\{C_\alpha\}$ in $\beta S$. It is immediate that in $\beta S$, $\overline{C_\alpha} = Z_\alpha$ so that $Z_\alpha$ is a zero set of $\beta S$. By the $C^*$-embedding of $S$ in $\beta S$, these zero sets are necessarily disjoint. An application of Theorem 4 completes the proof.

An important class of spaces that satisfy the conditions of the theorem are basically disconnected spaces.

**Corollary 6.** Let $S$ be a P-space. Then $S$ is $TC^*$-embedded in $\beta S$ and any family of $\omega_1$ disjoint zero sets of $S$ may be extended disjointly to any Tychonoff space $S$ is $C^*$-embedded in.

**Proof.** Since $S$ is a P-space, $\beta S$ is basically disconnected and each zero set of $S$ is a regular closed set.

**4. Some examples.** The author is indebted to E. van Douwen for pointing out the following theorem of Juhasz which will be useful in analyzing examples.

**Theorem A (Juhasz [8]).** If $X$ is compact (even if $X$ is the union of $G_\delta$-sets of some compactification, i.e. $X$ is of point-countable type) and $S$ has cellularity $m$, $m$ infinite, then $X$ has at most $2^m$ disjoint $G_\delta$-sets.
In the above theorem, we can replace $G_δ$-sets by zero sets and compact by pseudocompact sets based on Theorem 2.

We have already noted R. Fox's result that $R$ is not $TC^*$-embedded in $βR$. However as all disjoint families of regular zero sets are countable, these families are disjointly extendable. By Theorem 5, using CH, all disjoint families of compact zero sets of $R$ are disjointly extendable.

The following example of R. Pol, pointed out to the author by R. Hodel, shows that even in the case of dense $C^*$-embedding we may be limited in the cardinality of the number of disjoint zero sets that are extended in case $S$ is not pseudocompact or a $P$-space.

**Example 1 (R. Pol [9]).** The space $S$ is Tychonoff, satisfies c.c.c., has countable pseudocharacter and has cardinality $2^c$. Thus $S$ has $2^c$ disjoint zero sets. Furthermore $βS$ satisfies c.c.c; so by Theorem A the $z$-cellularity of $βS$ is a most $c$. So $S$ is not $TC^*$-embedded in $βS$.

**Example 2.** Let $X$ be the absolute of $S$; then $βX$ is the absolute of $βS$ [11] and since c.c.c. is preserved under taking absolutes [11], $X$ satisfies c.c.c. and since there is a continuous map from $X$ to $S$, $z(x) ≥ 2^c$. By Theorem A and the fact that $βX$ satisfies c.c.c., the $z$-cellularity of $βX$ is at most $c$. Thus $X$ is not $TC^*$-embedded in $βX$ and $X$ is extremally disconnected.

**Example 3.** Since $βN \sim N$ is a zero set of $βN$, $βN \sim N$ is $TC^*$-embedded in $βN$ and is not $T_2$-embedded in $βN$. Even though $C^*$-embedding implies $z$-embedding $TC^*$-embedding does not imply $T_2$-embedding.

Blair [2] has shown that if a weakly perfectly normal space is not realcompact one can add a point of $νX \sim X$ and obtain a weakly perfectly normal space that is not perfectly normal (the additional point is not a $G_δ$ in the space). We will call this space $W(X)$. It is clear that if $X$ is normal and hereditarily extremally disconnected then so will $W(X)$ be.

**Example 4 (Blair [2]).** If $D$ is a discrete space of measurable cardinality $W(D)$ is a hereditarily extremally disconnected normal space that is not perfectly normal. Furthermore it is easily shown that every subset of $W(D)$ is $TC^*$-embedded so that every subset of a space may be $TC^*$-embedded without the space being perfectly normal.

Below we have another example of this where we use the set-theoretic axiom club.

**Example 5 (Wage [10]).** The space $X$ is perfectly normal and extremally disconnected and constructed using club. Then $W(X)$ is normal and hereditarily extremally disconnected and is not perfectly normal.

The question then arises as to whether normal hereditarily extremally disconnected spaces have the property that every subset is $TC^*$-embedded, particularly under MA + $\sim$ CH.

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**References**


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