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EMBEDDING COSMIC SPACES IN LUSIN SPACES

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ABSTRACT. We show that every regular cosmic space can be embedded in a Lusin space. This answers a question posed by J. P. R. Christensen.

In his book [**2**], J. P. R. Christensen asks the following question: Can a regular cosmic space be embedded in an analytic space?

The purpose of this note is to give a positive answer to that question. The answer was also obtained by Gary Gruenhage and by Calbrix [**1**].

For undefined terminology we refer the reader to [**4**]. By regular we mean T_3. A T_0 space is called *cosmic* if it is a continuous image of a separable metric space. Michael [**7**] defined cosmic spaces and proved the following theorem.

THEOREM 0. *A T_0 space is cosmic if and only if it has a countable network.*

A *network* for a space X is a family $\mathcal{N}$ of subsets of X (not necessarily open) such that for every $x \in X$ and open U containing x there is an $N \in \mathcal{N}$ with $x \in N \subset U$.

A T_2 space X is called *analytic* if it is a continuous image of a complete separable metric space. A T_2 space X is called *Lusin* if it is a one-to-one continuous image of a complete separable metric space.

A centered system is a family of sets with the finite intersection property, and a centered system on a family $\mathcal{N}$ is a centered system whose members belong to $\mathcal{N}$. A centered system $\mathcal{F}$ of subsets of a space X converges to a point x of X if every neighbourhood of x contains an element of $\mathcal{F}$. Let $\mathcal{N} \subset \mathcal{P}(X)$, then for $A \subset X$ define $[A]_{\mathcal{N}} = \{x \in X$: there is a maximal centered system $\mathcal{F}$ on $\mathcal{N}$ which contains A such that $\mathcal{F}$ converges to $x\}$.

A family $\mathcal{N} \subset \mathcal{P}(X)$ is convergent if every maximal centered system $\mathcal{F}$ on $\mathcal{N}$ converges to some point x such that for any neighbourhood U of x there is an $A \in \mathcal{F}$ such that $[A]_{\mathcal{N}} \subset U$.

THEOREM 1. *A T_2 space X is analytic if and only if it has a countable convergent network.*

PROOF. Suppose $\mathcal{N} = \{N_n : n \in \omega\}$ is such a network. Without loss of generality we may assume that $\mathcal{N}$ is closed under finite intersections. For $x \in X$ let $\mathcal{F}_x$ denote a maximal centered system on $\mathcal{N}$ such that $\{N \in \mathcal{N} : x \in N\} \subset \mathcal{F}_x$. Choose $f_x \in {}^\omega\omega$ by the rules

$$f_x(n) = n \quad \text{if } N_n \in \mathcal{F}_x,$$

otherwise

$$f_x(n) = \min\{k \colon N_k \cap N_n = \emptyset \& N_k \in \mathcal{F}_x\}.$$

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The closure M of $\{f_x\colon x \in X\}$ in ${}^\omega\omega$ is a complete separable metric space. If $f \in M$, then $\{N_{f(i)}\colon i \in \omega\}$ is a centered system on $\mathcal{N}$ and there is a maximal centered system $\mathcal{F}_f$ extending it. By assumption, $\mathcal{F}_f$ converges to say $x_f \in X$. We claim the map $f \mapsto x_f$ is continuous. Thus X is analytic, for the fact that X is T_2 assures us that $x_{f_x} = x$, and thus the map is onto.

To prove the claim, suppose $x_f \in U$ which is open in X. There is an n with $x_f \in [N_n]_{\mathcal{N}} \subset U$ and $N_n \in \mathcal{F}_f$. If $f(n) = i \neq n$ then there is $x \in X$ with $f(n) = f_x(n)$ so $N_i \cap N_n = \varnothing$, a contradiction. So $\{g \in M\colon g(n) = n\}$ is an open set in M containing f and we claim that $\{x_g\colon g(n) = n\} \subset U$, proving that $f \mapsto x_f$ is continuous. But $\mathcal{F}_g$ converges to x_g and $N_n \in \mathcal{F}_g$ so $x_g \in [N_n]_{\mathcal{N}}$.

Going the other way assume $f\colon M \to X$ is continuous for some complete separable metric space M and T_2 space X. For each $x \in X$ choose $x' \in f^{-1}(x)$ and let M' be the closure in M of $\{x'\colon x \in X\}$. For each $n \in \omega$ choose a star finite cover $\mathcal{B}_n$ of M' by closed sets of diameter $< 1/2^n$. Let $\mathcal{B} = \bigcup_{n\in\omega} \mathcal{B}_n$. For $B \in \mathcal{B}$ let $B' = \{x' \in B\colon x \in X\}$ and $\mathcal{N} = \{f(B')\colon B \in \mathcal{B}\}$.

1°. $\mathcal{N}$ is a network for X: If $x \in U$ which is open in X, choose $B \in \mathcal{B}$ with $x' \in B \subset f^{-1}(U)$. Then $f(B') \subset U$ so $\mathcal{N}$ is a network.

2°. $\mathcal{N}$ is convergent: If $\mathcal{F}$ is a maximal centered system on $\mathcal{N}$ then $\{B\colon f(B') \in \mathcal{F}\}$ is a centered system and there is a (unique) $p \in \bigcap\{B\colon f(B') \in \mathcal{F}\}$ since each $B \in \mathcal{B}_i$ has diameter $< 1/2^i$ and meets only finitely many other members of $\mathcal{B}_i$ and the metric is complete. We claim that $\mathcal{F}$ converges to $f(p)$ and that for any neighbourhood U of $f(p)$ there is an $A \in \mathcal{F}$ so that $[A]_{\mathcal{N}} \subset U$. To prove that, choose $B \subset f^{-1}(U)$ with $f(B') \in \mathcal{F}$. If $x \in [f(B')]_{\mathcal{N}}$ there is $q \in M$ so that $q \in B$ and $f(q) = x$ so $[f(B')]_{\mathcal{N}} \subset f(B) \subset U$.

The inner characterization of regular analytic spaces is a bit simpler.

COROLLARY. *A regular space X is analytic if and only if it has a countable network $\mathcal{N}$ such that every maximal centered system on $\mathcal{N}$ converges.*

REMARK. Let us mention that Theorem 1 remains true if in the definitions before it one writes infinite instead of maximal. This new version of Theorem 1 seems to be more useful. For example, Hurewicz proved: A metrizable analytic space Y is σ-compact if and only if it does not contain a closed copy of the irrationals [**5**, p. 100], in a rather indirect way. One can use the new version of Theorem 1 to give a short alternative direct proof. In fact, that proof only requires Y to be regular (cf. also [**3**, Lemma 8.8] for another direct proof of the above).

From Theorem 1 one can easily derive an answer to Christensen's question, but we can prove a bit more.

Let us call a network $\mathcal{N}$ *complemented* if $X - N$ is the union of members of $\mathcal{N}$ for every $N \in \mathcal{N}$.

THEOREM 2. *A T_2 space X is Lusin if and only if X has a countable complemented network $\mathcal{N}$ such that every centered system on $\mathcal{N}$ has an intersection in X.*

PROOF. Suppose that $f\colon M \to X$ is continuous and one-to-one for some complete separable metric space M and T_2 space X. For each $n \in \omega$ choose a countable star finite closed cover $\mathcal{B}_n$ of M by sets of diameter $< 1/2^n$. Let $\mathcal{B} = \bigcup_{n\in\omega} \mathcal{B}_n$ and $\mathcal{N} = \{f(B)\colon B \in \mathcal{B}\}$. Since f is continuous and one-to-one and $\mathcal{B}$ is complemented,

$\mathcal{N}$ is a complemented network. If $\mathcal{F}$ is a centered system on $\mathcal{N}$ then $\mathcal{F}' = \{B \in \mathcal{B}\colon f(B) \in \mathcal{F}\}$ is also centered and there is a $y \in \bigcap \mathcal{F}'$. But $f(y) \in \bigcap \mathcal{F}$. Thus $\mathcal{N}$ is as desired.

Going the other way, assume that $\mathcal{N} = \{N_n\colon n \in \omega\}$ is a complemented network for X and every centered system on $\mathcal{N}$ has a nonempty intersection. If $x \in X$, define $f_x \in {}^\omega\omega$ by the rules

$$f_x(n) = n \quad \text{if } x \in N_n \qquad \text{and} \qquad f_x(n) = \min\{i\colon N_i \cap N_n = \varnothing \& x \in N_i\} \quad \text{if } x \notin N_n.$$

Since $\mathcal{N}$ is complemented this is possible.

Suppose $M = \{f_x\colon x \in X\}$ and $f \in \overline{M}$. Then $\{N_{f(n)}\colon n \in \omega\}$ is a centered system on $\mathcal{N}$ and, by assumption, there is $x \in \bigcap_{n\in\omega} N_{f(n)}$. We claim that $f = f_x$. For suppose $f(n) \neq f_x(n)$ for some n if $j = f_x(n)$, there is $y \in X$ with $f_y(n) = f(n)$ and $f_y(j) = f(j)$. Since $x \in N_j \cap N_{f(n)}$ j and $f(n)$ are minimal in $\{i\colon N_i \cap N_j = \varnothing\}$ for $x \in N_j$ and $y \in N_{f(j)}$ respectively. Since $x \in N_{f(j)}$, $j < f(j)$ and $y \notin N_j$. Thus $f_x(j) = j$ and $f_y(j) \neq j$; hence $N_{f_x(j)} \cap N_{f(j)} = \varnothing$ but this is impossible since $x \in N_{f_x(j)} \cap N_{f(j)}$.

Hence $f = f_x$, M is closed, and the map $f_x \mapsto x$ of M onto X is one-to-one.

This map is also continuous since if $x \in U$, open in X, $x \in N_n \subset U$ and $\{f \in M\colon f(n) = n\}$ is an open set in M, containing f_x, which is mapped into U. Thus X is Lusin.

COROLLARY. *If X is a regular cosmic space, X can be embedded in a Lusin space.*

PROOF. Because of the regularity of X we can shoose a closed network $\mathcal{N} = \{N_n : n \in \omega\}$ for X. Without loss of generality we may assume that $\mathcal{N}$ is closed under finite intersections. Let X' be the space consisting of all maximal centered systems on $\mathcal{N}$. If U is open in X let

$$U' = \{x' \in X'\colon \exists N \in \mathcal{N}(N \subset U \& N \in x')\}.$$

Every regular cosmic space is normal so sets U' form a base for a T_2 topology on X'. If $x \in X$, let $x^m = \{N \in \mathcal{N} : x \in N\}$. As $\mathcal{N}$ is a closed network x^m is a maximal centered system on $\mathcal{N}$. It is easy to check that the mapping $f\colon X \to X'$ given by $f(x) = x^m$ is an embedding.

For $N \in \mathcal{N}$ define $N' = \{x' \in X'\colon N \in x'\}$, and let $\mathcal{N}' = \{N'_n\colon n \in \omega\}$. If $\mathcal{F}'$ is a centered system on $\mathcal{N}'$ then $\{N_n\colon N'_n \in \mathcal{F}'\}$ is a centered system on $\mathcal{N}$ so there is $x' \in \bigcap \mathcal{F}'$. If $x' \notin N'_n$ there is an $N_m \in x'$ so that $N_m \cap N_n = \varnothing$. So $N'_m \cap N'_n = \varnothing$, hence $\mathcal{N}'$ is a complemented network and so X' is a Lusin space.

Let us note that the assumption of regularity of X is necessary, as shown in [**6**], and also that the Lusin space X' is, in general, unlikely to be regular, so the question whether or not a regular cosmic space can be embedded in a regular analytic space is still open.

In closing, we would like to mention an open question involving cosmic spaces.

If ${}^\omega X$ is hereditarily Lindelöf and hereditarily sequentially separable, is X cosmic? (A set $D \subset X$ is sequentially dense in X if for every $x \in X$ there is a sequence of points of D converging to x. A space X is sequentially separable if there is a countable sequentially dense $D \subset X$.)

Michael gives an example in [**8**] which shows that under CH the answer is no; Rudin [**9**] has proved that under CH there is a subset of the real line with the half-open-interval topology which is an example.

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