CARDINALITIES OF FIRST COUNTABLE \( R \)-CLOSED SPACES

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Abstract. It is now well known that first countable compact Hausdorff spaces are either countable or have cardinality \( c \). The situation for first countable \( H \)-closed spaces is that they have cardinality less than or equal to \( c \), and it is at least consistent that they may have cardinality \( \aleph_1 < c \). We show that the situation is quite different for first countable \( R \)-closed spaces. We begin by constructing an example which has cardinality \( \aleph_1 \). Let \( \lambda_0 \) be the smallest cardinal greater than \( c \) which is not a successor. For each cardinal \( \kappa \) with \( c \leq \kappa \leq \lambda_0 \) we construct a first countable \( R \)-closed space of cardinality \( \kappa \). We also construct a first countable \( R \)-closed space of cardinality \( \lambda_0 \). This seems to indicate that there is no reasonable upper bound to the cardinalities of \( R \)-closed spaces as a function of their character.

1. Introduction. A regular space \( X \) is \( R \)-closed if \( X \) is closed in every regular space containing \( X \) as a subspace, \( X \) is minimal regular if \( X \) has no coarser regular topology, and \( X \) is strongly minimal regular (SMR) if \( X \) has a base for the closed sets consisting of \( R \)-closed subsets. It is known that if \( X \) is SMR, then \( X \) is minimal regular and if \( X \) is minimal regular, then \( X \) is \( R \)-closed. A regular filter \( \mathcal{F} \) on a space \( X \) is a filter with the property that for \( A \in \mathcal{F} \), there is an open \( U \in \mathcal{F} \) such that \( \text{cl } U \subseteq A \). We shall assume that a regular filter consists of open sets. A filter \( \mathcal{F} \) is free if \( \bigcap \{ \text{cl } A : A \in \mathcal{F} \} = \emptyset \). An equivalent characterization of \( R \)-closed spaces is that there are no free regular filters, cf. \([BPS]\).

The class of \( R \)-closed spaces does not seem to be a very well behaved class. For instance, not every regular space can be densely embedded in an \( R \)-closed space \([He]\), and the product of \( R \)-closed spaces need not be \( R \)-closed \([P]\). These results are in sharp contrast to the classes of \( H \)-closed spaces and compact Hausdorff spaces, both of which are productive \([BPS]\), and every Hausdorff (Tychonoff) space can be densely embedded in a \( H \)-closed (compact Hausdorff) space. One of the few positive results about \( R \)-closed spaces is in \([DPI]\) where we show that every regular space can be embedded as a closed subspace of an \( R \)-closed space. This paper exhibits further distinctions between the class of \( R \)-closed spaces and the classes of compact Hausdorff spaces and \( H \)-closed spaces.

In the second section we construct an example of a first countable SMR space of cardinality \( \aleph_1 \). A space is \( \aleph_0 \)-bounded if every countable set has compact closure. Hechler \([H]\) has given consistent examples of first countable separable SMR spaces of cardinality \( \aleph_1 \). However, our example requires no special set theoretic assumptions and our example is \( \aleph_0 \)-bounded. In the final section we recursively construct
first countable SMR spaces for each cardinal \( \kappa \) with \( c \leq \kappa \leq \lambda_0 \) and a first countable SMR space of cardinality \( \lambda_0^\omega \).

A construction of \( R \)-closed spaces that has been put to good use by many authors is a technique due to Jones. In [DP1] we investigate this technique in some detail, but we will need a modification of it due to Stephenson [S]. We thank the referee for his or her useful suggestions.

1.1. Suppose that \( X \) is a regular space containing pairwise disjoint closed sets \( H_1, H_2 \) and \( K \). Suppose further that we have homeomorphisms \( f_1, f_2 : H_i \rightarrow K \). Let \( E \) be the equivalence relation on \( X \times \mathbb{Z} \), where \( \mathbb{Z} \) is the integers, defined by the rule \((x, i) \sim (y, j)\) if: (i) \( x = y \) and \( i = j \); (ii) \( x = f_2(y) \) and \( i + 1 = j \); (iii) \( y = f_2(x) \) and \( j + 1 = i \); (iv) \( x = f_1(y) \) and \( j + 1 = i \); (v) \( y = f_1(x) \) and \( i + 1 = j \); (vi) \( f_1(x) = f_2(y) \) and \( i + 2 = j \); or (vii) \( f_1(y) = f_2(x) \) and \( j + 2 = i \). Choose two new points \( \{P_+, P_-\} \) and let \( DJ(X) = \{P_+, P_\} \cup (X \times \mathbb{Z})/E \) and \( J(X) = \{P_+\} \cup (X \times \mathbb{N})/E_{|X \times \mathbb{N}|} \), where \( \mathbb{N} \) is the positive integers. For each \( n \in \mathbb{N} \) let \( V_n = \{P_+\} \cup X \times \{m \in \mathbb{Z} : m \geq n\} \) and \( W_n = \{P_\} \cup X \times \{m \in \mathbb{Z} : m \leq -n\} \). We topologize \( DJ(X) \) by giving \((X \times \mathbb{Z})/E \) the quotient topology and letting \( \{V_n : n \in \mathbb{N}\} \) and \( \{W_n : n \in \mathbb{N}\} \) form neighborhood bases for \( P_+ \) and \( P_- \) respectively. With this topology \( DJ(X) \) is a regular space, and we let \( J(X) \) have the subspace topology.

**Proposition 1.2.** Let \( X \) be locally compact and Hausdorff, and let \( H_1, H_2, K, f_1 \) and \( f_2 \) be as in 1.1. Suppose that for some \( n \in \omega \) and any sequence \( U_0, \ldots, U_{n+1} \) with \( \text{cl} \ U_{n+1} \subseteq \text{int} \text{cl} U_i \) for each \( i \leq n \), \( \text{cl} U_{n+1} \) not compact implies that \( \text{cl} U_0 \cap K \) is not compact. Then \( DJ(X) \) is SMR and \( J(X) \) is \( R \)-closed.

**Proof.** Since \( X \) is locally compact, so is \( DJ(X) \setminus \{P_+, P_-\} \). By 1.2 of [DP1] and the fact that for each \( m \in \mathbb{N} \), \( V_m \) and \( W_m \) are homeomorphic to \( J(X) \), it suffices to show that \( J(X) \) is \( R \)-closed.

Suppose that \( \mathcal{F} \) is a free regular filter on \( J(X) \). Since \( \mathcal{F} \) is free, there is a \( U_0 \in \mathcal{F} \) such that \( P_+ \notin \text{cl} U_0 \). Suppose that \( k \in \mathbb{N} \) is such that \( V_k \cap \text{cl} U_0 = \emptyset \). Let \( m = k \cdot (n + 1) \) and recursively choose \( \{U_i : i \leq n\} \subseteq \mathcal{F} \) such that \( \text{cl} U_{i+1} \subseteq U_i \) for \( i = 0, 1, \ldots, m - 1 \). Since \( \mathcal{F} \) is free, for some \( i < m \), \( \text{cl} U_m \cap X \times \{i\} \) is not compact. Therefore, by assumption on \( X \), \( \text{cl} U_{m-(n+1)} \cap K \times \{i\} \) is not compact. Since \( K \times \{i\} \) is identified with \( H_2 \times \{i + 1\} \) by \( f_2 \), we have that \( \text{cl} U_{m-(n+1)} \cap X \times \{i + 1\} \) is not compact. By induction, this implies that \( \text{cl} U_{m-n} \cap X \times \{k\} \) is not compact; however, this contradicts that \( \text{cl} U_0 \cap V_k = \emptyset \).

2. The \( \mathbb{N}_1 \) example. Let us begin by establishing some notation. Let \( \Lambda = \{\lambda_\alpha : \alpha < \omega_1\} \) be an order preserving listing of the limit ordinals less than \( \omega_1 \). For each \( n \in \omega \), let \( A_n = [(\omega_1 \setminus \Lambda) \times (n + 1) \cup \Lambda] \times \{n\} \), and let \( X = \bigcup_{n \in \omega} A_n \). Recall that an ordinal is the set of its predecessors. For each \( \alpha \in \omega_1 \), let

\[
X_\alpha = \bigcup_{n \in \omega} [\lambda_\alpha \setminus \Lambda \times (n + 1) \cup \{\lambda_\beta : \beta \leq \alpha\}] \times \{n\}.
\]

We will construct, for \( \alpha \in \omega_1 \), a topology \( \tau_\alpha \) on \( X_\alpha \) which is compact Hausdorff and has a basis of compact open sets. This construction is a modification of Vaughan’s [V] construction of a first countable, countably compact, nonnormal
space and is indeed itself such a space. We have not been able to show that Vaughan’s space satisfies the hypothesis of 1.2 and suspect it does not as it contains countably many pairwise disjoint clopen copies of $\omega_1$, whose union is dense.

To help clarify the construction we give an informal description first. Each column $A_n$ is $\omega_1 \times (n + 1)$ with the limit ordinals identified, and will be homeomorphic to $\omega_1$. We think of each column $A_n \setminus (\Lambda \times \{n\})$ as containing $n + 1$ columns. If we take a horizontal cross section at level $a \in \omega_1 \setminus \Lambda$ by choosing the element out of the $k$th column of $A_n$ for each $n > k$ this sequence will converge to the point in the $k$th column of $A_k$. The limit levels will compactify what is below them with, in addition, $\{(\lambda, n): 1 \leq n < \omega\}$ converging to $(\lambda, 0)$.

Suppose that $\alpha \in \omega_1$ and for $\beta < \alpha$ we have constructed a Hausdorff topology $\tau_\beta$ on $X_\beta$ such that

(a) $\tau_\beta$ has a countable base of compact open sets;
(b) for each $n \in \omega$, $\gamma \in \lambda_\beta \setminus \Lambda$ and $k < n$ the point $(\gamma, k, n)$ is isolated;
(c) for each $k \in \omega$ and $\gamma \in \lambda_\beta \setminus \Lambda$, a neighborhood base for $(\gamma, k, k)$ is $\{(\gamma, k, k)\} \cup \{(\gamma, k, m): m > n\}: k \leq n < \omega$;
(d) for each $\delta < \beta$ and $n < \omega$ the set $\{(\gamma, k, n): \gamma < \lambda_\beta, k \leq n\}$ is compact;
(e) for each $\delta < \beta$, $\{(\lambda_\beta, n): 0 < n < \omega\}$ converges to $(\lambda_\beta, 0)$; and
(f) $(X_\beta, \tau_\beta)$ is an open subspace of $(X_\beta', \tau_\beta')$ for each $\delta < \beta$.

We will define $\tau_\alpha$ by defining neighborhood bases at each of the points of $X_\alpha \setminus \bigcup_{\beta < \alpha} X_\beta$.

Step 1. For each $\gamma \in \lambda_\alpha \setminus \bigcup_{\beta < \alpha} \lambda_\beta + 1$ and $k < n < \omega$, we put $\{(\gamma, k, n)\}$ in $\tau_\alpha$.

Step 2. For each $\gamma \in \lambda_\alpha \setminus \bigcup_{\beta < \alpha} \lambda_\beta + 1$ and $k < \omega$, a neighborhood base for $(\gamma, k, k)$ will be $\{(\gamma, k, k)\} \cup \{(\gamma, k, m): m > n\}: k \leq n < \omega$.

Step 3. Let $\{\gamma_i: i \in \omega\} \subset \lambda_\alpha \setminus \Lambda$ be a strictly increasing sequence converging to $\alpha$.

For each $0 < n < \omega$, let $B_\delta(n) = [(\gamma_0 \setminus \Lambda) \times (n + 1) \cup (\Lambda \cap \gamma_0)] \times \{n\}$ and recursively define, for $1 \leq i < \omega$,

$$B_i(n) = [(\gamma_i \setminus \Lambda) \times (n + 1) \cup (\Lambda \cap \gamma_i)] \times \{n\} \setminus B_{i-1}(n).$$

It is easily seen that for each $n \in \omega \setminus \{0\}$, $\{B_i(n): i \in \omega\}$ is locally finite in $X_\alpha \setminus ((\lambda_\alpha) \times \omega)$. In addition, by induction, assumptions (c), (d), and (f), each $B_i(n)$ is a compact set. Each point of $B_i(n)$ has a neighborhood base of compact open sets already defined. By the definition of these neighborhood bases we may choose, for each $n > 0$, a locally finite collection of compact open subsets, $\{C_i(n): i \in \omega\}$, of $X_\alpha \setminus ((\lambda_\alpha) \times \omega)$ so that $C_i(n) \cap A_n = B_i(n)$ and $C_j(n) \cap A_j = \emptyset$ for $j \leq \max\{i, n\}$, $j \neq n$. Recursively define for $1 \leq n < \omega$ a neighborhood base at $(\alpha, n)$ as follows: For each $j < \omega$,

$$W_j((\alpha, n)) = [\{(\alpha, n)\} \cup \{C_i(n): j < i < \omega\}] \setminus \{W_n((\alpha, m)): 0 < m < n\}.$$
Step 4. By construction $X_\alpha \backslash \{(\lambda_\alpha, 0)\}$ has a countable base of compact open sets. We let $X$ be the one point compactification of $X_\alpha \backslash \{(\lambda_\alpha, 0)\}$. This clearly satisfies conditions (e) and (f) and the others obviously hold by construction.

We let $\tau$ be the topology on $X$ generated by the open base $\bigcup_{\alpha < \omega_1} \tau_\alpha$. With this topology $X$ is first countable, zero dimensional (has a base of clopen sets), locally compact, $\aleph_0$-bounded, nonnormal and has cardinality $\aleph_1$. To see that $X$ is $\aleph_0$-bounded we simply note that any countable set is contained in some $X_\alpha$. Nonnormality will be a trivial consequence of the following.

Let $U$ be an open set containing a closed noncompact subset of $A_n$, where $n \in \omega$. We will show that $\text{cl } U \cap A_0$ is not compact. Since $A_n$ is homeomorphic to $\omega_1$, $A_n \backslash U$ is countable [GJ]. Therefore there is an $\alpha < \omega_1$ such that for $\gamma > \alpha$ and $\gamma \notin \lambda$, $(\gamma, n, n) \in U$. For each such $\gamma$, there is a $k \in \omega$ such that $\{(\gamma, n, j) : j > k\} \subset U$ since $U$ is open. It follows that there are an uncountable set $\Gamma \subset \omega_1 \backslash \lambda$ and a $k \in \omega$ such that for each $\gamma \in \Gamma$, $\{(\gamma, n, j) : j > k\} \subset U$. Now, for each limit $\lambda$ of $\Gamma$ and $j > k$, $(\lambda, j) \in \text{cl } U$ since $(\lambda, j) \in \text{cl}\{((\gamma, n, j)) : \gamma \in \Gamma\}$. Therefore $(\lambda, 0) \in \text{cl } U$ for each limit $\lambda$ of $\Gamma$. Since $\Gamma$ is uncountable, $\text{cl } U \cap A_0$ is not compact.

Now since each $A_n$ is homeomorphic to $\omega_1$, if we let $H_1 = A_1$, $H_2 = A_2$ and $K = A_0$, we have that $X$ satisfies the conditions of 1.2 with $n = 0$. We have shown

**Theorem 2.1.** $DJ(X)$ is a first countable SMR space with cardinality $\aleph_1$.

3. Cardinals greater than $\mathfrak{c}$. In this section we will construct first countable SMR spaces with cardinalities up to $\aleph_0$ where $\aleph_0$ is the first singular cardinal greater than $\mathfrak{c}$. Assuming GCH, $\aleph_0 = \aleph_\omega$. We will make extensive use of a space developed by Mrowka and Isbell then modified by Stephenson [S] and Hechler [H].

3.1. Suppose $D$ is an infinite discrete set and $R$ is a family of infinite pairwise almost disjoint countable subsets of $D$. ($A$ and $B$ are almost disjoint if $|A \cap B| < \omega$.) We will give $X = D \cup R$ the following topology. $D$ is open and discrete while a neighborhood base for $r \in R$ is $\{r \cup r' : r' \text{ is a finite subset of } r\}$. It is easily shown that $X$ is a first countable locally compact Hausdorff space.

Suppose that $R$ contains disjoint subsets $H, K$ with $H \cup K = R$ and $|H| = |K|$. Suppose further that there is an $n \in \omega$ such that for any sequence $U_0, \ldots, U_{n+1} \subset D$ with $\text{cl } U_{i+1} \subset \text{cl } U_i$ for $i < n$, $|\text{cl } U_{n+1} \cap H| \geq \aleph_0$ implies $|\text{cl } U_0 \cap K| \geq \aleph_0$. It is then clear that if we let $H_1 \cup H_2 = H$ with $|H_1| = |H_2|$ and $f_i$ be any isomorphism from $H_i$ to $K$ for $i = 1, 2$, then $X$ satisfies the conditions 1.2. Therefore $DJ(X)$ would be SMR. This will be our plan, to recursively construct $X_\kappa$, for $c \leq \kappa \leq \aleph_0$, so that $X_\kappa = D_\kappa \cup R_\kappa$ with $|R_\kappa| = \kappa$ and $R_\kappa$ satisfying the above.

To begin we borrow a result by Hechler [H] whose purpose was to remove the assumption of the continuum hypothesis from an example by Stephenson [S].

3.2. Theorem. There is a maximal almost disjoint family $R_c$ of infinite subsets of $\omega$ such that $|R_c| = c$ and there is a subset $H_c \subset R_c$ with $|H_c| = c$ such that for $U \subset \omega$, $|\{r \in H_c : U \cap r| = \aleph_0\} \geq \aleph_0$ implies $|\{r \in R_c \setminus H_c : U \cap r| = \aleph_0\} = c$.

For a space $X = D \cup R$ as in 3.1 we will say that $(X, H, K)$ satisfies $(\ast_c)$ if the following hold: (i) $H \subset R$ and $K = R \setminus H$; (ii) $|H| = |K| = |R|$; and (iii) for
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Let \( U_0, \ldots, U_{n+1} \subseteq D \) with \( \{U_i \} \subseteq \text{int} \{U_i \} \) for \( i \leq n \), then \( |\{U_{n+1} \cap H\}| \geq \aleph_0 \) implies \( |\{U_0 \cap K\}| \geq \aleph_0 \). Let \( D_\omega = \omega \) and let \( R_\omega \) and \( H_\omega \) be as in 3.2. Then \( X_\omega = D_\omega \cup R_\omega \) with the topology as in 3.1 satisfies \((\ast_0)\). Indeed, for \( r \in R_\omega \) and \( W \subseteq D_\omega \), \( r \in \text{cl} \{W\} \) if \( r \cap W = \emptyset \). Let \( K_\omega = R_\omega / H_\omega \).

Therefore if \( U_0, U_1 \subseteq D \) are as in (iii) with \( |U_0 \cap H_\omega| \geq \aleph_0 \) then by 3.2 \( |U_0 \cap K_\omega| = \omega \). Hence \( |U_0 \cap K_\omega| \geq \aleph_0 \).

3.3. The main construction. Let \( \kappa \) be a cardinal such that \( \kappa^\omega = \kappa \). Suppose that \( X_\kappa = D_\kappa \cup R_\kappa \) is as in 3.1 with \( |D_\kappa| < |R_\kappa| = \kappa \). Suppose also that \( n \in \omega \) and that \( (X_\kappa, H_\kappa, K_\kappa) \) satisfy \((\ast_\kappa)\). We shall let \( D_{\kappa^+} = D_\kappa \times \kappa^+ \) and define an almost disjoint family, \( R_{\kappa^+} \), of countable subsets of \( D_{\kappa^+} \) and subsets \( H_{\kappa^+} \) and \( K_{\kappa^+} \) of \( R_{\kappa^+} \) so that \( (X_{\kappa^+}, H_{\kappa^+}, K_{\kappa^+}) \) satisfies \((\ast_{\kappa^+})\).

An ordinal \( \delta \) is said to have countable cofinality, \( \text{cf} \delta = \omega \), if there is a sequence \( \{\gamma_n: n \in \omega\} \subseteq \delta \) such that \( \delta = \sup(\gamma_n) \). For each \( \delta < \kappa^+ \) with \( \text{cf} \delta = \omega \) we shall construct a set \( R(\delta) \) as follows. Choose any strictly increasing sequence \( \{\gamma_n: n \in \omega\} \) converging to \( \delta \). Let \( R(\delta) \) be a maximal family of almost disjoint uncountable countable subsets of \( D_\delta \times \delta \) such that for each \( r \in R(\delta) \) and \( n \in \omega \), \( |r \cap D_\delta \times \gamma_n| \leq \aleph_0 \). Note that \( |R(\delta)| \leq |D_\delta \times \delta| = \kappa^\omega = \kappa \). Fix an injection \( f_\delta \) from \( R(\delta) \) into \( H_\delta \). We define \( R_{\kappa^+} = \{r \cup (f_\delta(r) \times \{\delta\}): \delta < \kappa^+ \} \) with \( \text{cf} \delta = \omega \) and \( r \in R(\delta) \) \( \cup \{r \times \{\gamma\}: \gamma < \kappa^+ \} \). Let \( K_{\kappa^+} = D_{\kappa^+} \setminus \{r \times \{\gamma\}: \gamma < \kappa^+ \} \), \( r \in R_{\kappa^+} \), and if \( \text{cf} \gamma = \omega \), \( r \notin f_\delta(\gamma) \). Each element of \( R_{\kappa^+} \) is a countable subset of \( D_{\kappa^+} \) and \( K_{\kappa^+} \) is an almost disjoint family. Let us define \( H_{\kappa^+} = \{r \in R_{\kappa^+}: \text{there is an } r_1 \in H_\delta \text{ with } r \cup \{r_1\} \subseteq r \} \).

Suppose first that there is a \( \gamma < \kappa^+ \) such that \( \{r \subseteq U_{n+2} \cap H_\kappa: \text{there is an } r_1 \in H_\delta \text{ with } r \cup \{r_1\} \subseteq r \} \) is infinite. Therefore, since \( (X_\kappa, H_\kappa, K_\kappa) \) satisfies \((\ast_\kappa)\), \( \{r \subseteq U_{n+2} \cap H_\kappa: \text{there is an } r_1 \in H_\delta \text{ with } r \cup \{r_1\} \subseteq r \} \) is uncountable. Hence \( |U_0 \cap K_{\kappa^+}| \geq \aleph_0 \). On the other hand, suppose that there is an infinite increasing sequence \( \{\gamma_j: j \in \omega\} \subseteq \kappa^+ \) such that, for each \( j \in \omega \), there are an \( r_j \in H_\kappa \) and an \( r(j) \times \{\gamma_j\} \subseteq r(j) \times \kappa^+ \). Therefore, for each \( j \in \omega \), \( U_{n+2} \cap D_\kappa \times (\gamma_j + 1) \) is infinite. So let \( \delta = \sup(\gamma_j: j \in \omega) \) and choose an infinite set \( S \subseteq R(\delta) \) such that \( s \cup \{f_\delta(s) \times \{\delta\}\} \subseteq \{r \subseteq U_{n+2} \cap H_\kappa: \text{there is an } r_1 \in H_\delta \text{ with } r \cup \{r_1\} \subseteq r \} \) for each \( s \in S \). This can be done since \( s \subseteq D_\kappa \times \gamma_j \) is finite for each \( j \in \omega \) and \( s \subseteq R(\delta) \). Therefore, \( s \cup \{f_\delta(s) \times \{\delta\}\} \subseteq \{r \subseteq U_{n+2} \cap H_\kappa: \text{there is an } r_1 \in H_\delta \text{ with } r \cup \{r_1\} \subseteq r \} \) for each \( s \in S \).

We now obtain the following results.

**Theorem 3.4.** For each cardinal \( \kappa \) with \( c \leq \kappa < \lambda_0 \) there is a first countable SMR space with cardinality \( \kappa \).

**Proof.** Observe that for a cardinal \( \kappa \) with \( \text{cf}(\kappa) > \omega \), \( \kappa^\omega = \sum(\alpha^\omega: \alpha < \kappa) \). It follows that \( (c^+)^\omega = \sum(\alpha^\omega: \alpha < c^+) = c^+ \cdot c = c^+ \). Similarly, by induction, \( \kappa^\omega = \kappa \) for \( c < \kappa < \lambda_0 \). Therefore, by 3.2 and 3.3, we have a space \( X_\kappa \) of cardinality \( \kappa \) such that \( (X_\kappa, H_\kappa, K_\kappa) \) satisfies \((\ast_\kappa)\) for some \( n \in \omega \). We partition \( H_\kappa \) into two sets \( H_1 \) and \( H_2 \) with \( |H_1| = |H_2| \). By 1.2, \( DJ(X_\kappa) \) is a first countable SMR space.

The following result follows easily from the results in \([P]\).
Theorem 3.5. A product of first countable SMR spaces is itself SMR.

Corollary 3.6. There are first countable, SMR spaces of cardinality $\lambda_0$ and $\lambda^c_0$, respectively.

Proof. For $n \in \omega$, let $\kappa_n$ be the $n$th successor of $c$. Thus, $\lambda_0 = \Sigma\{\kappa_n : n \in \omega\}$. For each $n \in \omega$, let $X_n$ be a first countable, SMR space of cardinality $\kappa_n$. Let $Y$ be the topological sum of $\{X_n : n \in \omega\}$ (we write $Y = \bigcup \{X_n : n \in \omega\}$) and $X = \{P_+\} \cup Y$. Now, $U \subseteq X$ is defined to be open if $U \cap Y$ is open in $Y$ and if $P_+ \subseteq U$, there is $m \in \omega$ such that $U \supseteq \bigcup \{X_n : n \geq m\}$. It is immediate that $X$ is first countable and $R$-closed and has cardinality $\lambda_0$. Since each $X_n$ is SMR, then $X$ is also SMR. Finally let $Z = \prod\{X_n : n \in \omega\}$. By 3.5, $Z$ is SMR and is clearly first countable. The cardinality of $Z$ is $|\prod\{X_n : n \in \omega\}| = \lambda^c_0$.

References


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