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A CHARACTERIZATION OF UNIFORM PARACOMPACTNESS

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ABSTRACT. Main result: a uniform space X is uniformly paracompact [**R**] iff for some (and then any) compactification K of X and for any compact $C \subset K \setminus X$ closed disjoint sets $X \times C$ and the diagonal $\Delta_X (= \{\langle x, x\rangle | x \in X\})$ can be separated by a uniformly continuous function on the semiuniform product $X * K$.

H. Tamano gave [**T**] the following interesting result. For a T_1-completely regular space X the following conditions are equivalent:

(a) X is paracompact,

(b) $X \times \beta X$ is normal, i.e. disjoint closed sets can be separated by a continuous function,

(c) for each compact $C \subset \beta X \setminus X$ the two closed sets $X \times C$ and the diagonal $\Delta_X = \{\langle x, x\rangle | x \in X\}$ can be separated by a continuous function on $X \times \beta X$.

There are several ways to generalize this result to uniform spaces. Separation of closed sets in the usual (i.e. categorical) product by various types of functions has been considered by Z. Frolík in [$\mathbf{F_2}$]; in this case (b) is always equivalent to (c). The same author studied separation of closed sets by coz-functions if the product is interpreted as the semiuniform product $*$ (in the sense of Isbell [$\mathbf{I_1}$, III, p. 44]), (b) and (c) are not equivalent in this situation, indeed, they give characterizations of two distinct classes of uniform spaces [$\mathbf{F_3}$].

Here we study the case when the product is $*$ and separating functions are uniformly continuous. Since the properties studied do not depend on the compactification we use (by a compactification of a uniform space we mean a compactification of the underlying topological space), the Samuel compactification is just one of compactifications. Definition of $*$ and properties we need are recalled in §1.

If not stated otherwise, by a space we mean uniform T_2-space. Following M. D. Rice [**R**], a space X is called uniformly paracompact if every open cover of X has a uniformly locally finite open refinement. Again, we recall the necessary facts in §1, more detailed information can be found in [$\mathbf{H_1}$].

The reader is referred to Isbell's book [$\mathbf{I_1}$] for definitions and basic facts concerning uniform spaces.

We can now formulate our results.

THEOREM 1. *A space X is uniformly paracompact if and only if for some, and then any, compactification K of X the following holds*:

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for each compact $C \subset K \setminus X$ *there exists a uniformly continuous function* f *on* $X * K$ *such that* f *is* 1 *on* Δ_X *and* f *is* 0 *on* $X \times C$.

THEOREM 2. *A space* X *is fine and paracompact if and only if for some, and then any, compactification* K *of* X *the following holds*:

for each pair F_0, F_1 *of closed disjoint sets on* $X \times K$ *there exists a uniformly continuous function* f *on* $X * K$ *such that* f *is* 0 *on* F_0, *and* f *is* 1 *on* F_1.

COROLLARY. *The following conditions are equivalent*:

(1) X *is supercomplete*,

(2) *for some, and then any, compactification* K *of* X *the following holds*:

for each compact $C \subset K \setminus X$ *there exists a uniformly continuous function* f *on* $\lambda X * K$ *such that* f *is* 1 *on* Δ_X *and* f *is* 0 *on* $X \times C$,

(3) *for some, and then any, compactification* K *of* X *the following holds*:

for each pair of closed disjoint sets F_0, F_1 *in* $X \times K$ *there exists a uniformly continuous function* f *on* $\lambda X * K$ *such that* f *is* 1 *on* F_1 *and* f *is* 0 *on* F_0.

1. Definitions and facts. Following Isbell, we denote the semiuniform product of spaces X and Y by $X * Y$. The underlying set of $X * Y$ coincides with the underlying set of $X \times Y$. The uniformity of $X * Y$ is the weak uniformity induced on $X \times Y$ by all semiuniform mappings onto, mapping f on $X \times Y$ being semiuniform if $\{f(-, y) \mid y \in Y\}$ is an equiuniformly continuous family on X and each $f(x, -)$ is uniformly continuous on Y. It is clear that $X * Y$ is homeomorphic to $X \times Y$ and that all semiuniform mappings on $X \times Y$ are uniformly continuous on $X * Y$.

We recall a few facts.

Fact 1 [**I**$_1$, III 23]. Every uniform covering of $X * Y$ has a refinement of the form

$$\{U_\alpha \times V_\beta^\alpha\}_{\alpha \in A, \beta \in B_\alpha} \tag{s}$$

where $\{U_\alpha\}$ and $\{V_\beta^\alpha\}_{\beta \in B_\alpha}$ are uniform coverings of X and Y, respectively.

Fact 2 [**I**$_1$, VII 29]. If X is fine and Y is compact, then $X * Y$ is fine.

Fact 3. If X has a point-finite basis, then for any Y, every covering of the form (s) is uniform on $X * Y$. Proof of this statement is almost the same as the proof of the fact that the first Ginsburg-Isbell derivative of X is a uniformity [**I**$_1$, VII 5].

Fact 4 [**H**$_1$]. A space X is uniformly paracompact if and only if for some, and then any, compactification K of X and for any compact $C \subset K \setminus X$ there exists a uniform cover $\{V_a\}$ of X such that for each V_a, $\bar{V}_a^K \cap C = \varnothing$.

We shall also need the easy observation mentioned by A. Hohti [**H**$_1$].

Fact 5. A uniformly paracompact space has a point-finite basis. We give the proof for the reader's convenience.

Suppose that $\mathcal{W}$ is a uniform covering of a uniformly paracompact space X. Take an open uniform covering $\mathcal{V}$, $\mathcal{V} \overset{*}{\prec} \mathcal{U}$. $\mathcal{V}$ has an open refinement $\mathcal{G}$ locally finite relative to $\{\mathrm{St}(W, \mathcal{W}) \mid W \in \mathcal{W}\}$, $\mathcal{W}$ being a uniform covering of X, $\mathcal{W} \prec \mathcal{V}$. Hence $\{\mathrm{St}(G, \mathcal{W}) \mid G \in \mathcal{G}\}$ is the desired uniform refinement of $\mathcal{U}$.

2. Proofs of theorems.

PROOF OF THEOREM 1. Suppose that K is a compactification of X, let $X \times C$ and Δ_X be separated by a uniformly continuous function for any compact $C \subset K \times X$. Separating the function provides us a covering $\mathcal{U} = \{U_\alpha \times V_\beta^\alpha\}$ (of the form (s)) such that $\mathrm{St}(\Delta_X, \mathcal{U}) \cap X \times C = \varnothing$. Suppose that $c \in \overline{U}_\alpha^K \cap C$ for some α. Then there exists β such that V_β^α is a neighborhood of c. Thus $V_\beta^\alpha \cap U_\alpha \neq \varnothing$ and $U_\alpha \times V_\beta^\alpha$ meets both sets Δ_X and $X \times C$ which is impossible. Hence, $\overline{U}_\alpha \cap C = \varnothing$ for each α. According to Fact 4, X is uniformly paracompact.

Let X be uniformly paracompact, K be a compactification of X, $C \subset K \setminus X$ be a compact set. Then there exists a uniform covering $\mathcal{V} = \{V_a\}$ such that $\overline{V}_a^K \cap c = \varnothing$ for each a. We may and shall assume that $\mathcal{V}$ is point-finite. Obviously, it is enough to construct a uniform cover $\mathcal{W}$ of $X * K$ such that $\mathrm{St}(\Delta_X, \mathcal{W}) \cap X \times C = \varnothing$. Since $\overline{V}_a^K \cap c = \varnothing$, we can choose a binary open cover $\{U_a^1, U_a^2\}$ of K such that $V_a \subset U_a^1$ and $V_a \cap U_a^2 = \varnothing$. Put

$$\mathcal{W} = \left\{ V_a \times U_a^i \middle| V_a \in \mathcal{V}, i = 1,2 \right\}.$$

PROOF OF THEOREM 2. $X * K$ is fine and paracompact, provided X is fine and paracompact. So "only if" is obvious.

Suppose that disjoint closed sets can be separated. Then X is paracompact. Let $\{U_a\}$ be an open cover of X and let $\{V_\alpha\}$ be its closed locally finite refinement. For each α, take one a_α such that $V_\alpha \subset U_{a_\alpha}$. Then $\{U_{a_\alpha}\} \prec \{U_a\}$. Then Δ_X and $T = \bigcup \left\{ V_\alpha \times K - U_{a_\alpha} \right\}$ are closed disjoint sets. Existence of separating functions provides us a cover of the form (s), $\mathcal{W} = \{U_\delta \times V_\gamma^\delta\}$, such that $\mathrm{St}(\Delta_X, \mathcal{U}) \cap T = \varnothing$. Let $z \in U_\delta \cap V_\alpha$. Suppose $U_\delta \not\subset U_{a_\alpha}$, i.e. there exists $x \in U_\delta \setminus U_{a_\alpha}$. Take V_x^δ such that $x \in V_x^\delta$. Hence $U_\delta \times V_\gamma^\delta \cap \Delta_X \neq \varnothing \neq U_\delta \times V_\gamma^\delta \cap T$ which is impossible.

We recall that X is supercomplete if its hyperspace $H(X)$ of closed subsets (see [$\mathbf{I_1}$]) is complete, which is equivalent [$\mathbf{I_2}$] to λX, is fine and paracompact.

PROOF OF COROLLARY. We know that (1) → (3) → (2). (2) says that λX is uniformly paracompact, hence $\lambda(\lambda X) = \lambda X$ is fine (λ of a uniformly paracompact space is fine [**R**]) and paracompact.

The equivalence of (1) and (3) has been proved by A. Hohti [$\mathbf{H_2}$] by a different technique. In fact, he uses separation in $\lambda(X \times K)$, but $\lambda(X \times K) = \lambda X * K$.

CONCLUDING REMARK. We recall that $A \subset X$ is a cozero set in X if there exists a uniformly continuous function f on X such that $A = \{x \mid f(x) \neq 0\}$. A real-valued function is called coz-function if the preimage of every open set is a cozero set. Denote by mX the metric-fine coreflection of X (see [$\mathbf{F_1}$] for the description of mX). In [$\mathbf{F_3}$] it is proved that for compact K,

$$mX * K = m(X * K).$$

Since coz-functions on X coincide with uniformly continuous functions on mX, the application of our results to mX yields the following parts of two theorems from [$\mathbf{F_3}$].

THEOREM 1$'$. *The following conditions are equivalent*:

(i) *mX is uniformly paracompact*,

(ii) *for some* (*and then any*) *compactification K of X and for any compact $C \subset K \setminus X$ there exists a coz-function g on $X * K$ such that $g|\Delta_X = 1$ and $g|X \times C = 0$*.

THEOREM 2$'$. *The following conditions are equivalent*:

(i) *mX is fine and paracompact*,

(ii) *for some* (*and then any*) *compactification K of X and for each pair F_0, F_1 of closed disjoint sets there exists a coz-function on $X * K$ such that $g|F_0 = 0$ and $g|F_1 = 1$*.

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