PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 89, Number 3, November 1983

A NOTE ON THE STABLE HOMOTOPY GROUPS OF $MSp(n)$

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ABSTRACT. The kernel of the epimorphism $\pi^S_{8n+3}(MSp(n)) \to \pi_{4n+3}(MSp)$ is a cyclic group of order $8m(n+1)$ for some integer $m(n+1)$ defined using the characteristic numbers of the symplectic cobordism classes, and this epimorphism splits for some n.

1. Introduction and results. Let $MSp(n)$ be the Thom space of the universal bundle over $BSp(n)$, and $MSp = \{MSp(n), \varepsilon_n\}$ be the Thom spectrum of the symplectic cobordism theory, where ε_n: $\Sigma^4 MSp(n) \to MSp(n+1)$ is the structure map. Let $\varepsilon = \varepsilon(n, N)$: $\Sigma^{4N} MSp(n) \to MSp(n+N)$ $(N > n \geqslant 1)$ be the composition of $\Sigma^{4(N-j-1)}\varepsilon_{n+j}$ for $N > j \geqslant 0$. Then ε_*: $\pi_i^S(MSp(n)) \to \pi_{i-4n}(MSp)$ is an isomorphism for $i \leqslant 8n+2$ and an epimorphism for $i = 8n+3$, where we identify $\pi_{i+4N}(\Sigma^{4N}MSp(n))$ and $\pi_{i+4N}(MSp(n+N))$ with $\pi_i^S(MSp(n))$ and $\pi_{i-4n}(MSp)$ respectively.

Let $P_k \in H^{4k}(BSp)$ $(k \geqslant 1)$ be the kth symplectic Pontrjagin class and $P_k[u]$ be the characteristic number of $u \in \pi_{4k}(MSp)$. Then $P_k[u]$ is divisible by 8 for any $u \in \pi_{4k}(MSp)$ (cf. [**1**, Theorem I]). Put

$$m(k) = \text{g.c.d.}\left\{\tfrac{1}{8}P_k[u] \colon u \in \pi_{4k}(MSp)\right\}.$$

Then we have the following

THEOREM 1. (i) *The kernel of the epimorphism* ε_*: $\pi^S_{8n+3}(MSp(n)) \to \pi_{4n+3}(MSp)$ *is a cyclic group of order* $8m(n+1)$.

(ii) *Assume that* (a) $m(n+1)$ *is odd or* (b) $2\pi_{4n+3}(MSp) = 0$. *Then* ε_* *in* (i) *is split epimorphic, that is,*

$$\pi^S_{8n+3}(MSp(n)) \cong Z_{8m(n+1)} \oplus \pi_{4n+3}(MSp).$$

In the cases $n = 1, 2$, these facts are contained within the results due to S. O. Kochman and V. P. Snaith [**3**]. The assumption (b) in the theorem is valid for $n \leqslant 8$ by D. M. Segal [**6**] and for $n < 26$ by the announcement of S. O. Kochman [**2**]. The assumption (a) is valid for some n's by [**1**, Theorem II(iv)], and we have the following corollary.

COROLLARY 2. *If* $n = 2^k + 2^l - 1$ *or* $2^k + 2^l - 2$ $(k, l \geqslant 0)$, *then* $\pi^S_{8n+3}(MSp(n))$ *is isomorphic to* $Z_8 \oplus \pi_{4n+3}(MSp)$ *for* $n \neq 2$, *and to* $Z_8 \oplus Z_3$ *for* $n = 2$.

Received by the editors October 6, 1982 and, in revised form, February 1, 1983.
1980 *Mathematics Subject Classification.* Primary 57R20, 57R90, 55N22, 55Q10.
Key words and phrases. MSp, $MSp(n)$, stable homotopy group, characteristic number, fibration.
[1]The author wishes to express his thanks to the referee for his kind suggestions.

The corresponding results on the complex cobordism theory were obtained by E. Rees and E. Thomas [5].

The author thanks K. Morisugi for his valuable suggestion.

2. Proof of the theorem. We assume $n \geqslant 2$. Convert $\varepsilon = \varepsilon(n, N)$ into a fibration with fiber $F(n)$ for $N > n$

$$F(n) \xrightarrow{j} \Sigma^{4N} MSp(n) \xrightarrow{\varepsilon} MSp(n+N). \tag{2.1}$$

Then $F(n)$ is $(4N + 8n + 2)$-connected, and a part of the homotopy exact sequence for the fibration is given as follows:

$$\begin{aligned} \pi_{4n+4}(MSp) &\xrightarrow{\partial} \pi_{4N+8n+3}(F(n)) \xrightarrow{j_*} \pi^S_{8n+3}(MSp(n)) \\ &\xrightarrow{\varepsilon_*} \pi_{4n+3}(MSp) \to 0, \end{aligned} \tag{2.2}$$

where we identify $\pi_{4(n+N)+i}(MSp(n+N))$ with $\pi_i(MSp)$ $(i = 4n+3, 4n+4)$. Let τ: $H^i(F(n)) \to H^{i+1}(MSp(n+N))$ be the transgression of the fibration (2.1). Then, for $i = 4N + 8n + 3$, τ is isomorphic onto the subgroup generated by UP_{n+1}, where UP_{n+1} is the image of P_{n+1} under the Thom isomorphism U: $H^{4(n+1)}(BSp(n+N)) \to H^{4(N+2n+1)}(MSp(n+N))$. Hence $H^{4N+8n+3}(F_n) = Z$, and its generator w satisfies

$$\tau(w) = UP_{n+1}. \tag{2.3}$$

Let $\bar{w} \in H_{4N+8n+3}(F(n))$ be the dual homology class to w, and $\iota \in \pi_{4N+8n+3}(F(n)) = Z$ be the generator which satisfies $H(\iota) = \bar{w}$ for the Hurewicz isomorphism H.

Proof of Theorem 1(i). By (2.2) and the definition of $m(n+1)$, we have only to show

$$\partial u = P_{n+1}[u]\iota \quad \text{up to sign for any } u \in \pi_{4n+4}(MSp). \tag{2.4}$$

Consider the following diagram which commutes up to sign:

$$\begin{array}{ccccc} \pi_{4n+4}(MSp) & \cong & \pi_{4N+8n+4}(MSp(n+N)) & \xrightarrow{\partial} & \pi_{4N+8n+3}(F(n)) \\ \downarrow H & & \downarrow H & & \cong\downarrow H \\ H_{4n+4}(MSp) & \cong & H_{4N+8N+4}(MSp(n+N)) & \xrightarrow{\tau} & H_{4N+8n+3}(F(n)) \end{array}$$

Let $\partial u = k\iota$ for some integer k. Then, by this diagram and (2.3), we have $k = \langle H\partial u, w\rangle = \langle H(u), \tau(w)\rangle = \langle H(u), UP_{n+1}\rangle = P_{n+1}[u]$ up to sign, and (2.4) holds. Q.E.D.

Let b_n: $MSp(n) \to \Omega^{4N} MSp(n+N)$ be the adjoint map of ε, and i: $MSp(n) \to \Omega^{4N}\Sigma^{4N} MSp(n)$ be the natural inclusion. Convert these maps into fibrations. Then we have the homotopy commutative diagram

$$\begin{array}{ccccc} F' & = & F' & & \\ \downarrow & & \downarrow & & \\ F_n & \to & MSp(n) & \xrightarrow{b_n} & \Omega^{4N} MSp(n+N) \\ \downarrow q & & \downarrow i & & \\ \Omega^{4N} F(n) & \to & \Omega^{4N}\Sigma^{4N} MSp(n) & \xrightarrow{\Omega^{4N}\varepsilon} & \Omega^{4N} MSp(n+N) \end{array} \tag{2.5}$$

where q is the restriction of i, and F' and F_n are the homotopy fibers of i and b_n respectively.

The following lemma is proved in [**1**, Proposition 3.5].

LEMMA 2.6. $\pi_{8n+2}(F_n) = 0$.

We also have the following

LEMMA 2.7. $\pi_{8n+2}(F') = Z_8$.

PROOF. Apply the methods and results due to R. J. Milgram [**4**, Theorems 1.11, 3.7 and Propositions 3.1, 3.6], and use (2.5) and [**1**, Lemma 3.4]. Then we see the following:

(2.8) F' has the homotopy type of a cell complex

$$\left(S^{8n-1} \vee \left(S^{8n} \cup_2 e^{8n+1}\right) \vee S^{8n+2}\right) \cup \left(e^{8n+3} \vee e^{8n+3}\right) \cup e^{8n+4}$$

through dimension $8n + 3$.

(2.9) Let ϕ_1 and ϕ_2 be the attaching maps of $(8n + 3)$-cells in (2.8). Then, by taking a suitable basis in

$$\pi_{8n+2}\left(S^{8n-1} \vee \left(S^{8n} \cup_2 e^{8n+1}\right) \vee S^{8n+2}\right) = Z_{24} \oplus Z_4 \oplus Z,$$

$\phi_1 = (1, 2k, 0)$ and $\phi_2 = (6l, 1, 2)$ hold for some $k \in Z_4$, $l \in Z_{24}$. The desired result follows immediately from these facts. Q.E.D.

PROOF OF THEOREM 1(ii). Consider the following commutative diagram of two exact sequences derived from the fibrations in (2.5):

$$(2.10)\qquad \begin{array}{ccccccc} \pi_{4N+8n+3}(F(n)) & \xrightarrow{j_*} & \pi^S_{8n+3}(MSp(n)) & \xrightarrow{\varepsilon_*} & \pi_{4n+3}(MSp) & \to 0 \\ \| & & \downarrow \partial' & & & \\ Z & \xrightarrow{\partial} & \pi_{8n+2}(F') = Z_8 & \to & \pi_{8n+2}(F_n) = 0 & \end{array}$$

where ∂' and ∂ are the connecting homomorphisms. If the assumption (a) of the theorem holds, then ∂' induces an isomorphism between the 2-components of Im j_* and $\pi_{8n+2}(F')$, because Im $j_* = Z_{8m(n+1)}$ by (i) of the theorem. Since $\pi_{4n+3}(MSp)$ is a 2-torsion group, we have a splitting for the inclusion Im $j_* \to \pi^S_{8n+3}(MSp(n))$, and the theorem holds. Next we assume (b). In (2.10), $\partial \otimes 1\colon Z \otimes Z_2 \to Z_8 \otimes Z_2$ is an isomorphism, and $j_* \otimes 1\colon\ Z \otimes Z_2 \to \pi^S_{8n+3}(MSp(n)) \otimes Z_2$ has the left inverse $(\partial \otimes 1)^{-1}(\partial' \otimes 1)$. Thus we have the exact sequence

$$0 \to Z_2 \to \pi^S_{8n+3}(MSp(n)) \otimes Z_2 \to \pi_{4n+3}(MSp) \to 0,$$

where $\pi_{4n+3}(MSp) \otimes Z_2 = \pi_{4n+3}(MSp)$ by the assumption. This fact implies that Im j_* is a direct summand of $\pi^S_{8n+3}(MSp(n))$, and we complete the proof. Q.E.D.

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