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IRREDUCIBLE 3-MANIFOLDS OF GENUS 3 CONTAINING A 2-SIDED PROJECTIVE PLANE

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ABSTRACT. It will be shown that the minimum genus over all Heegaard splittings which yield irreducible closed 3-manifolds containing a 2-sided projective plane but not homeomorphic to $P^2 \times S^1$ is equal to 3, with an infinite number of examples of genus 3.

1. Introduction. A 3-manifold is said to be *P^2-containing* if it contains a 2-sided projective plane. Since a projective plane is nonorientable, so is a P^2-containing 3-manifold. One of typical P^2-containing closed 3-manifolds is $P^2 \times S^1$ which is irreducible and which has Heegaard genus 2. (Hereafter Heegaard genus will be called simply "genus".) See Figure 2 for a genus 2 Heegaard splitting of $P^2 \times S^1$.

Recently, Ochiai [**10**] has proved that if a P^2-containing closed 3-manifold admits a Heegaard splitting of genus 2 then it is homeomorphic to $P^2 \times S^1$. So $P^2 \times S^1$ may be called the simplest P^2-containing 3-manifold in the sense of Heegaard splitting. On the other hand, there have been constructed an infinite number of P^2-containing, irreducible, closed 3-manifolds by Row [**11**], Negami [**9**] and so on. Their examples are complicated, however, and seem to have higher genus. From this situation, a natural question arises: What is the minimum genus over all Heegaard splittings which yield P^2-containing, irreducible, closed 3-manifolds not homeomorphic to $P^2 \times S^1$.

In this paper, we shall give the answer to this question:

THEOREM. *There exist an infinite number of P^2-containing, irreducible, closed 3-manifolds of genus* 3.

Ochiai's result implies that the minimum genus in question is greater than 2, and hence it is equal to 3.

2. Construction of examples. It is hardly possible to find P^2-containing, irreducible, closed 3-manifolds dealing with only *Heegaard diagrams*. Even if they were found, it would be very difficult to check the existence of 2-sided projective planes and their irreducibility. So we shall construct our examples $P(n)$ as results of Dehn surgeries on $P^2 \times S^1$ along a knot.

Figure 1 illustrates $P^2 \times S^1$ cut open into a 3-ball $D^2 \times I$ like a drum and the knot K with its tubular neighborhood $U(K)$ along which we carry out Dehn

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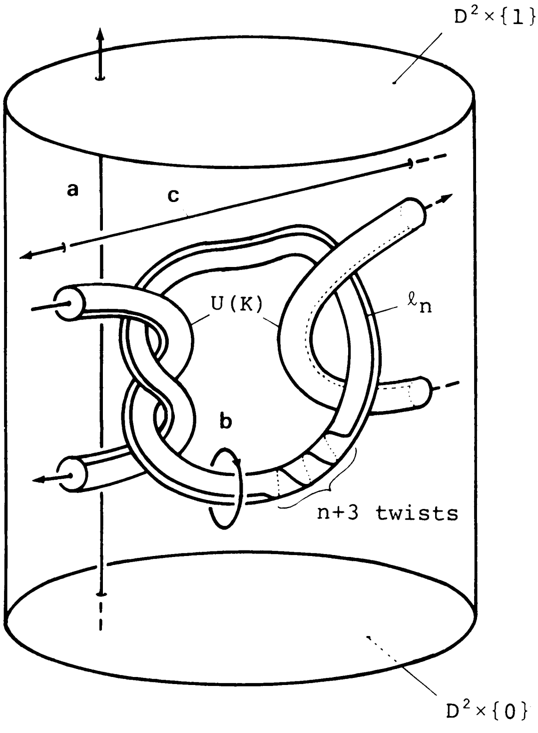

FIGURE 1

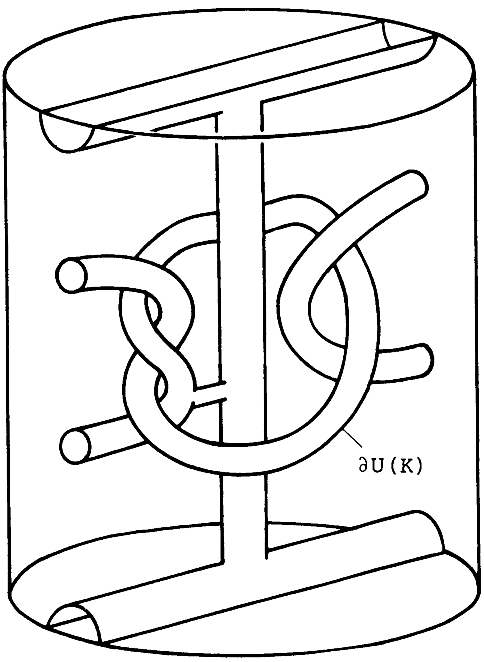

FIGURE 2

surgeries. Sew up $D^2 \times I$ along the annulus $\partial D^2 \times I$ levelwise by the antipodal maps on $\partial D^2 \times \{t\}$ ($t \in I$) and identify the resulting top and bottom projective plane, then $P^2 \times S^1$ will be obtained. The closed curve l_n indicated on $\partial U(K)$ is a surgery instruction for a nonnegative integer n. We shall remove the interior of the solid torus $U(K)$ from $P^2 \times S^1$ and attach $D^2 \times S^1$ to $\partial U(K)$ by a homeomorphism: $\partial D^2 \times S^1 \to \partial U(K)$ which sends $\partial D^2 \times \{*\}$ onto l_n. Let $P(n)$ denote the result of this Dehn surgery on $P^2 \times S^1$ ($n = 0, 1, 2, \ldots$).

We shall show the desired properties of $P(n)$ in order.

A. The 3-manifold $P(n)$ is P^2-containing: There is a fiber projective plane which does not meet $U(K)$ in $P^2 \times S^1$, say the quotient of $D^2 \times \partial I$, and it is naturally embedded in $P(n)$.

B. The 3-manifolds $P(n)$'s are homologically distinct from one another: Observe that $H_1(P^2 \times S^1 - K)$ is isomorphic to $\mathbf{Z} + \mathbf{Z} + \mathbf{Z}_2$ and is generated by three loops a, b, c shown in Figure 1. The two loops a, b generate the free part and the other c generates the torsion part of $H_1(P^2 \times S^1 - K)$. The surgery instruction l_n represents $n[b]$ since $[l_0] = 0$ in $H_1(P^2 \times S^1 - K)$. Thus $H_1(P(n))$ is isomorphic to $\mathbf{Z} + \mathbf{Z}_n + \mathbf{Z}_2$, and hence $H_1(P(n))$ and $H_1(P(m))$ are isomorphic if and only if $n = m$.

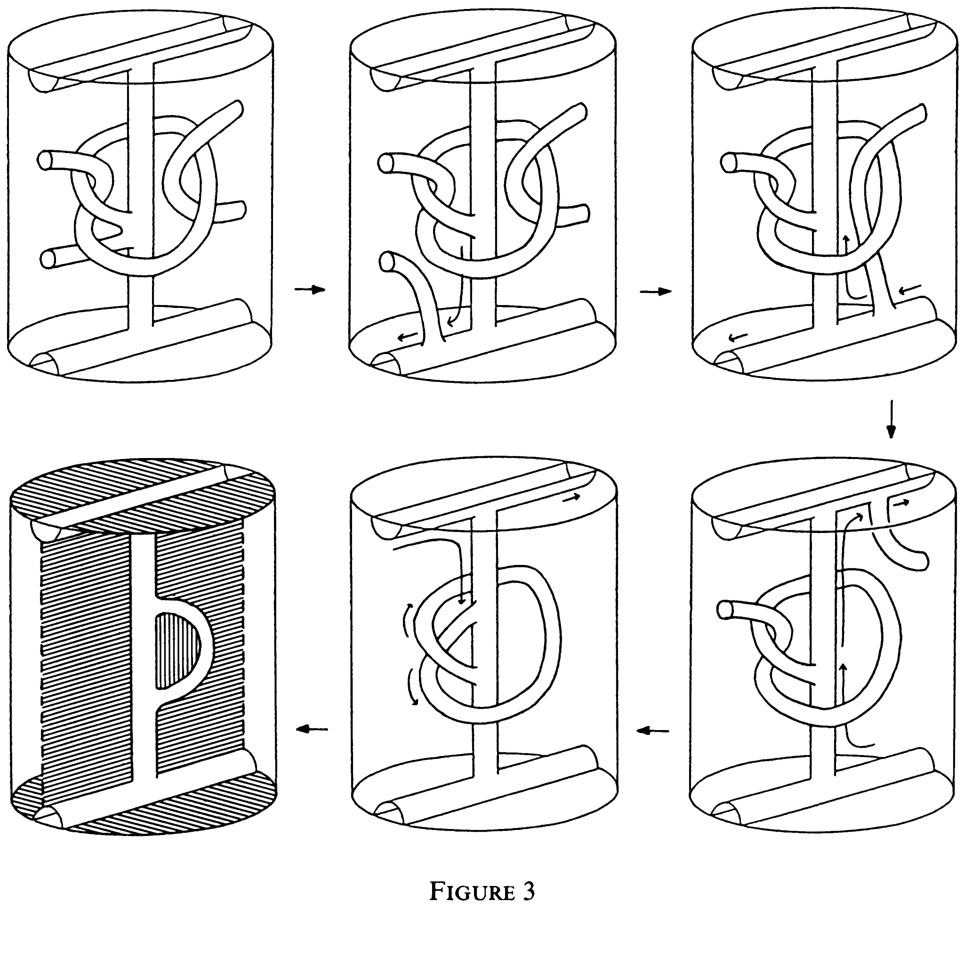

FIGURE 3

C. The genus of $P(n)$ is equal to 3: Figure 2 describes a Heegaard surface of genus 3 in $P(n)$. To see this, regard the surface as one in $P^2 \times S^1$ and transform it by an ambient isotopy of $P^2 \times S^1$ as illustrated in Figure 3. Three meridian disks of the "outer" region of the surface are indicated in the final stage of the consecutive figures. It is clear that its "inner" region is a handlebody of genus 3. (If you neglect the part of $\partial U(K)$ in Figure 2 then the surface can be regarded as a Heegaard surface of genus 2 in $P^2 \times S^1$.) So the genus of $P(n)$ does not exceed 3. On the other hand, $P(n)$ is homologically distinct from $P^2 \times S^1$ unless $n = 1$. Also $P(1)$ is not homeomorphic to $P^2 \times S^1$, which will be shown later with special treatment. Therefore $P(n)$ has no Heegaard splitting of genus 2 by Ochiai's result, and hence the genus of $P(n)$ is equal to 3.

D. The 3-manifold $P(n)$ is irreducible if n is even: The orientable double cover of $P^2 \times S^1$ is $S^2 \times S^1$. The knot K is lifted to a two-component link $\tilde{K}$ in $S^2 \times S^1$ which is τ-equivariant for the covering transformation τ of period 2. Cut $S^2 \times S^1$ along a fiber 2-sphere missing $\tilde{K}$ and cap off each 2-sphere boundary component with a 3-ball. Then S^3 will be obtained and the involution τ will induce an orientation-reversing involution σ on S^3 with two fixed points in the two added 3-balls. We shall consider $\tilde{K}$ as a σ-equivariant link in S^3. Let $R(n)$ be the result of the σ-equivariant Dehn surgery on S^3 along $\tilde{K}$ with one surgery coefficient n and the other $-n$, corresponding to the Dehn surgery along K for $P(n)$. Then the orientable double cover $\tilde{P}(n)$ of $P(n)$ decomposes into the connected sum $S^2 \times S^1 \# R(n)$.

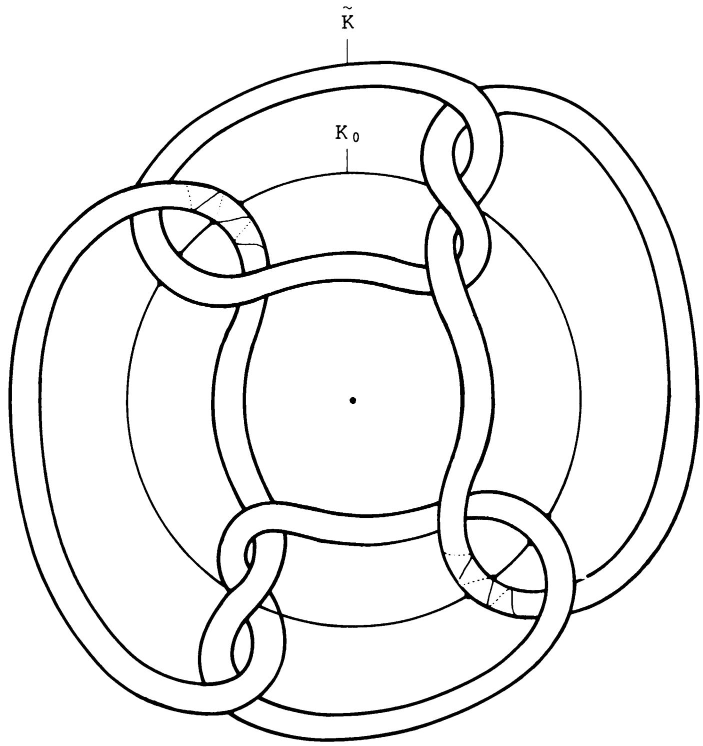

FIGURE 4

The link $\tilde{K}$ can be arranged to be in the position, as shown in Figure 4, where there is an orientation-preserving involution on S^3 with fixed point set a trivial knot K_0 which induces in each component of $\tilde{K}$ an involution with two fixed points. That is, $\tilde{K}$ is strongly-invertible. Therefore by results of Montesinos [**8**], $R(n)$ is a 2-fold branched cover of S^3. Figure 5 shows the branch set $L(n)$, which is a knot if n is odd and which is a link with three components K_1, K_2, K_3 if n is even. In the latter case, we have

$$\operatorname{lk}(K_1, K_2) = -\frac{n}{2} - 4, \quad \operatorname{lk}(K_2, K_3) = \frac{n}{2} + 4, \quad \operatorname{lk}(K_3, K_1) = 0,$$

where $\operatorname{lk}(K_i, K_j)$ is the linking number of K_i and K_j.

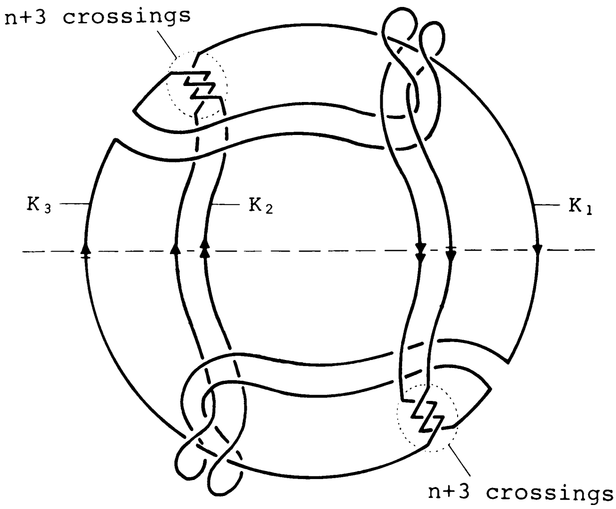

FIGURE 5

Now suppose that $P(n)$ were reducible, then $P(n)$ would decompose into a connected sum of $P^2 \times S^1$ and a lens space $L(n, q)$ because of the additivity of genus [2, 5]. The orientable double cover of $P^2 \times S^1 \# L(n, q)$ has a connected sum decomposition $S^2 \times S^1 \# L(n, q) \# L(n, q)$ in the unoriented sense, while it would decompose into $S^2 \times S^1 \# R(n)$, too. By the uniqueness of prime decompositions [7], $R(n)$ would have to decompose into $L(n, q) \# L(n, q)$. Bonahon [**1**] and Hodgson [**3**] have proved that a lens space $L(p, q)$ is a 2-fold branched cover of S^3 branched over a link L if and only if L has the same link type as a 2-bridge link $K(p, q)$. Combining it with Kim and Tollefson's result [**6**], we can show that $L(n)$ is a composition of two $K(n, q)$'s. When n is even, the composite link $K(n, q) \# K(n, q)$ has three components and the absolute value of the linking number of each pair of its components does not exceed $n/2$, contrary to the above observation on $\operatorname{lk}(K_i, K_j)$. Thus if n is even then $P(n)$ is irreducible.

REMARK. Even when n is odd, $P(n)$ will be irreducible if the primarity of the knot $L(n)$ is assured.

E. The 3-manifold $P(1)$ is an irreducible homology $P^2 \times S^1$ but is not homeomorphic to $P^2 \times S^1$: If $P(1)$ were homeomorphic to $P^2 \times S^1$ then $R(1)$ would be homeomorphic to S^3 and the knot $L(1)$ would be a trivial one [**12**]. However, the nontriviality of $L(1)$ can be shown by the algorithm proposed in [**4**], since $L(1)$ has a 3-bridge form (and other $L(n)$ do). The upper and lower halves of $L(n)$ in Figure 5 can be considered as over and under bridges in a 3-bridge form. Thus $P(1)$ is not homologically but is topologically distinct from $P^2 \times S^1$. The irreducibility of $P(1)$ follows from the fact that there is no homology 3-sphere of genus 1.

It has been shown that $P(2n)$'s justify the theorem.

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