COUNTABLE PRODUCTS
OF SCATTERED PARACOMPACT SPACES

MARY ELLEN RUDIN AND STEVE WATSON

Abstract. In this paper we prove that the product of countably many scattered
paracompact spaces is even ultraparacompact.

Telgársky [1] has shown that scattered paracompact spaces are ultraparacompact.
Verbally, H. Martin has asked if a product of countably many spaces with exactly
one nonisolated point has to be paracompact. We prove

Theorem. The product of countably many scattered paracompact spaces is ultra-
paracompact.

All spaces are assumed Hausdorff. A space is ultraparacompact if every open cover
has a disjoint open refinement. We occasionally use the word refinement when less
than the whole space is covered: if so the covered subspace is always mentioned. A
scattered space \( X \) is \( \bigcup_{\alpha < \lambda} X^\alpha \) for some minimal ordinal \( \lambda \) where, for \( \alpha < \lambda \), \( X^\alpha \) is the
set of all isolated points of \( X - \bigcup_{\beta < \alpha} X^\beta \). The order of \( X \) is \( \lambda \) and rank of \( x \in X \) is
the \( \alpha < \lambda \) with \( x \in X^\alpha \). We say a subset \( A \) of \( X \) is topped if \( A \) has a unique point of
maximal rank (i.e. the top of \( A \)). For completeness we prove

Lemma. Suppose \( \mathcal{G} \) is an open cover of a paracompact scattered space \( Y \). Then \( \mathcal{G} \) has
a disjoint, topped, open refinement (covering \( Y \)).

Proof. Suppose \( (\text{order } Y) \) is minimal for the lemma to fail.

Case (1). \( (\text{Order } Y) \) is a limit. There is a locally finite open refinement \( \mathcal{K} \) of \( \mathcal{G} \) by
sets whose closures have order less than \( (\text{order } Y) \). Let \( \mathcal{K} \) be a locally finite closed
refinement of \( \mathcal{K} \).

For \( H \in \mathcal{K} \), let \( K_H = \bigcup \{ K \subset H | K \in \mathcal{K} \} \). Since \( (\text{order } H) < (\text{order } Y) \), there is
a disjoint, open in \( H \), refinement \( \mathcal{G}_H \) of \( \{ H, H - K_H \} \) covering \( H \). Let \( \mathcal{G}_H = \{ J \in \mathcal{G}_H | J \cap K_H \neq \emptyset \} \).

Since \( \mathcal{G} = \bigcup_{H \in \mathcal{K}} \mathcal{G}_H \) is a locally finite cover of \( Y \) by clopen sets, by the standard
result of subtraction one can find an open, disjoint refinement \( \mathcal{E} \) of \( \mathcal{G} \) covering \( Y \). Since
(\text{order } L) < (\text{order } Y), each \( L \in \mathcal{E} \) can be covered by a set \( S_L \) of disjoint,
topped open sets. Thus \( \bigcup_{L \in \mathcal{E}} S_L \) is a disjoint, topped open refinement of \( \mathcal{G} \) as
desired.

Case (2). \( (\text{Order } Y) = \alpha + 1 \). Let \( Y^\alpha \) be the set of all points of \( Y \) of rank \( \alpha \). Since
\( Y^\alpha \) is a closed discrete subset of the paracompact \( Y \), there is a disjoint open
refinement $\mathcal{H}$ of $\mathcal{B}$ covering $Y^\alpha$ with each member of $\mathcal{H}$ containing precisely one point of $Y^\alpha$. Choose an open set $U$ with $Y^\alpha \subset U \subset \overline{U} \subset \bigcup \mathcal{H}$. Since $(\text{order } (Y - U)) < (\text{order } Y)$, there is a disjoint, topped, open in $Y - U$, refinement $\mathcal{H}$ of \( \{ \bigcup \mathcal{H} - U \} \cup \{ G - \overline{U} \mid G \in \mathcal{B} \} \) covering $Y - U$. Taking $\mathcal{G} = \{ K \in \mathcal{H} \mid K \cap \overline{U} = \emptyset \}$, $\mathcal{G} \cup \{ H - \bigcup \mathcal{G} \mid H \in \mathcal{H} \}$ is a disjoint, topped, open refinement of $\mathcal{B}$ covering $Y$ as desired.

The lemma is proved.

**Proof of the Theorem.** Suppose that for each $n \in \omega$, $X_n$ is a paracompact scattered space, $X = \prod_{n \in \omega} X_n$, and $\emptyset$ is an open cover of $X$.

Let $\Omega$ be the set of all subsets of $X$ which cannot be covered by any disjoint, open refinement (not necessarily covering $X$) of $\emptyset$. We make frequent use of: (*) If a member of $\Omega$ is the union of disjoint clopen sets, then one of these sets is in $\Omega$.

We assume $X \in \Omega$ in order to get a contradiction.

For each $i \in \omega$ we presently choose $k_i \in \omega$ and a function $f_i$ having domain $\omega$ such that $f_i(n)$ is a topped clopen subset of $X_n$ if $n < k_i$, $f_i(n) = X_n$ if $n \geq k_i$, and $\Pi_{n \in \omega} f_i(n) \in \Omega$.

Let $k_0 = 0$; thus each $f_0(n) = X_n$ and $\Pi_{n \in \omega} f_0(n) = X \in \Omega$.

Having defined $k_i$ and $f_i$ we consider two cases.

**Case (1).** For each $n < k_i$, there is a clopen $U_n$ in $X_n$ with $(\text{top } f_i(n)) \subseteq U_n$ and $(\Pi_{n < k_i} U_n \times \Pi_{n \geq k_i} X_n) \not\in \Omega$.

By (*), there is $m < k_i$ with at least $(f_i(m) - U_m) \times \Pi_{n \neq m} f(n) \in \Omega$. By the Lemma, there is a disjoint, topped, open cover $\mathcal{V}$ of $(f_i(m) - U_m)$. Define $k_{i+1} = k_i$, $f_{i+1}(n) = f_i(n)$ for $n \neq m$, and choose $f_i(m) \in \mathcal{V}$, by (*), so that $\Pi_{n \in \omega} f_{i+1}(n) \in \Omega$.

**Case (2).** Not Case (1). By the Lemma, there is a disjoint, topped, open cover $\mathcal{G}$ of $X_{k_i}$. Define $k_{i+1} = k_i + 1$, $f_{i+1}(n) = f_i(n)$ for $n \neq k_i$, and choose $f_{i+1}(k_i) \in \mathcal{G}$, by (*), so that $\Pi_{n \in \omega} f_{i+1}(n) \in \Omega$.

Since Case (1) implies $k_{i+1} = k_i$ and $\text{rank}(\text{top } f_{i+1}(m)) < \text{rank}(\text{top } f_i(m))$ for some $m < k_i$, and there is no infinite decreasing sequence of ordinals, Case (2) must hold for infinitely many $i \in \omega$.

Since Case (2) implies $k_{i+1} > k_i$, for every $n \in \omega$, there is $i_n \in \omega$ with $n < k_{i_n}$. Hence $f_{i_n}(n)$ has a top. Since $\text{rank}(\text{top } f_{i+1}(n)) \leq \text{rank}(\text{top } f_i(n))$ we can choose $i_n$ sufficiently large so that, for all $i \geq i_n$, $\text{rank}(\text{top } f_i(n)) = \text{rank}(\text{top } f_{i_n}(n))$. Thus, for $i \geq i_n$, $\text{top}(f_i(n)) = \text{top}(f_{i_n}(n))$.

If $t$ is the point of $X$ with $t(n) = \text{top}(f_i(n))$, then $t \in O \in \emptyset$. So there is $k \in \omega$ and for each $n < k$ a clopen $O_n$ in $X_n$ such that $t(n) \in O_n$ and $(\Pi_{n < k} O_n \times \Pi_{n \geq k} X_n) \subseteq O$.

If $i \geq i_n$ for all $n < k$, $n < k$ implies $n < k_i$ and $\text{top}(f_i(n)) = t(n)$. So regardless of $\text{top}(f_i(n))$ for $k < n < k_i$, Case (1) holds for $i$. This contradicts the fact that Case (2) holds infinitely often.

**References**


Department of Mathematics, University of Wisconsin, Madison, Wisconsin 53706