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COUNTABLE PRODUCTS OF SCATTERED PARACOMPACT SPACES

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ABSTRACT. In this paper we prove that the product of countably many scattered paracompact spaces is even ultraparacompact.

Telgársky [1] has shown that scattered paracompact spaces are ultraparacompact. Verbally, H. Martin has asked if a product of countably many spaces with exactly one nonisolated point has to be paracompact. We prove

THEOREM. *The product of countably many scattered paracompact spaces is ultraparacompact.*

All spaces are assumed Hausdorff. A space is *ultraparacompact* if every open cover has a disjoint open refinement. We occasionally use the word *refinement* when less than the whole space is covered: if so the covered subspace is always mentioned. A *scattered* space X is $\bigcup_{\alpha<\lambda} X^\alpha$ for some minimal ordinal λ where, for $\alpha < \lambda$, X^α is the set of all isolated points of $X - \bigcup_{\beta<\alpha} X^\beta$. The *order* of X is λ and *rank* of $x \in X$ is the $\alpha < \lambda$ with $x \in X^\alpha$. We say a subset A of X is *topped* if A has a unique point of maximal rank (i.e. the top of A). For completeness we prove

LEMMA. *Suppose $\mathcal{G}$ is an open cover of a paracompact scattered space Y. Then $\mathcal{G}$ has a disjoint, topped, open refinement (covering Y).*

PROOF. Suppose (order Y) is minimal for the lemma to fail.

Case (1). (Order Y) *is a limit*. There is a locally finite open refinement $\mathcal{H}$ of $\mathcal{G}$ by sets whose closures have order less than (order Y). Let $\mathcal{K}$ be a locally finite closed refinement of $\mathcal{H}$.

For $H \in \mathcal{H}$, let $K_H = \bigcup\{K \subset H \mid K \in \mathcal{K}\}$. Since (order $\overline{H}$) $<$ (order Y), there is a disjoint, open in $\overline{H}$, refinement $\mathcal{J}_H$ of $\{H, \overline{H} - K_H\}$ covering $\overline{H}$. Let $\mathcal{I}_H = \{J \in \mathcal{J}_H \mid J \cap K_H \neq \varnothing\}$.

Since $\mathcal{I} = \bigcup_{H\in\mathcal{H}} \mathcal{I}_H$ is a locally finite cover of Y by clopen sets, by the standard technique of subtraction one can find an open, *disjoint* refinement $\mathcal{L}$ of $\mathcal{I}$ covering Y. Since (order L) $<$ (order Y), each $L \in \mathcal{L}$ can be covered by a set $\mathcal{S}_L$ of disjoint, topped open sets. Thus $\bigcup_{L\in\mathcal{L}} \mathcal{S}_L$ is a disjoint, topped open refinement of $\mathcal{G}$ as desired.

Case (2). (*Order* Y) $= \alpha + 1$. Let Y^α be the set of all points of Y of rank α. Since Y^α is a closed discrete subset of the paracompact Y, there is a disjoint open

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refinement $\mathcal{H}$ of $\mathcal{G}$ covering Y^α with each member of $\mathcal{H}$ containing precisely one point of Y^α. Choose an open set U with $Y^\alpha \subset U \subset \overline{U} \subset \bigcup \mathcal{H}$. Since (order $(Y - U)) <$ (order Y), there is a disjoint, topped, open in $Y - U$, refinement $\mathcal{K}$ of $\{\bigcup \mathcal{H} - U\} \cup \{G - \overline{U} \mid G \in \mathcal{G}\}$ covering $Y - U$. Taking $\mathcal{J} = \{K \in \mathcal{K} \mid K \cap \overline{U} = \varnothing\}$, $\mathcal{J} \cup \{H - \bigcup \mathcal{J} \mid H \in \mathcal{H}\}$ is a disjoint, topped, open refinement of $\mathcal{G}$ covering Y as desired.

The lemma is proved.

Proof of the theorem. Suppose that for each $n \in \omega$, X_n is a paracompact scattered space, $X = \prod_{n \in \omega} X_n$, and $\mathcal{O}$ is an open cover of X.

Let Ω be the set of all subsets of X which cannot be covered by any disjoint, open refinement (not necessarily covering X) of $\mathcal{O}$. We make frequent use of: $(*)$ If a member of Ω is the union of disjoint clopen sets, then one of these sets is in Ω.

We assume $X \in \Omega$ in order to get a contradiction.

For each $i \in \omega$ we presently choose $k_i \in \omega$ and a function f_i having domain ω such that $f_i(n)$ is a topped clopen subset of X_n if $n < k_i$, $f_i(n) = X_n$ if $n \geq k_i$, and $\prod_{n \in \omega} f_i(n) \in \Omega$.

Let $k_0 = 0$; thus each $f_0(n) = X_n$ and $\prod_{n \in \omega} f_0(n) = X \in \Omega$.

Having defined k_i and f_i we consider two cases.

Case (1). For each $n < k_i$, there is a clopen U_n in X_n with $(\operatorname{top} f_i(n)) \in U_n$ and $(\prod_{n<k_i} U_n \times \prod_{n \geq k_i} X_n) \notin \Omega$.

By $(*)$, there is $m < k_i$ with at least $(f_i(m) - U_m) \times \prod_{n \neq m} f(n) \in \Omega$. By the Lemma, there is a disjoint, topped, open cover $\mathcal{V}$ of $f_i(m) - U_m$. Define $k_{i+1} = k_i$, $f_{i+1}(n) = f_i(n)$ for $n \neq m$, and choose $f_i(m) \in \mathcal{V}$, by $(*)$, so that $\prod_{n \in \omega} f_{i+1}(n) \in \Omega$.

Case (2). *Not Case* (1). By the Lemma, there is a disjoint, topped, open cover $\mathcal{U}$ of X_{k_i}. Define $k_{i+1} = k_i + 1$, $f_{i+1}(n) = f_i(n)$ for $n \neq k_i$, and choose $f_{i+1}(k_i) \in \mathcal{U}$, by $(*)$, so that $\prod_{n \in \omega} f_{i+1}(n) \in \Omega$.

Since Case (1) implies $k_{i+1} = k_i$ and $\operatorname{rank}(\operatorname{top} f_{i+1}(m)) < \operatorname{rank}(\operatorname{top} f_i(m))$ for some $m < k_i$, and there is no infinite decreasing sequence of ordinals, Case (2) must hold for infinitely many $i \in \omega$.

Since Case (2) implies $k_{i+1} > k_i$, for every $n \in \omega$, there is $i_n \in \omega$ with $n < k_{i_n}$. Hence $f_{i_n}(n)$ has a top. Since $\operatorname{rank}(\operatorname{top} f_{i+1}(n)) \leq \operatorname{rank}(\operatorname{top} f_i(n))$ we can choose i_n sufficiently large so that, for all $i \geq i_n$, $\operatorname{rank}(\operatorname{top} f_i(n)) = \operatorname{rank}(\operatorname{top} f_{i_n}(n))$. Thus, for $i \geq i_n$, $\operatorname{top}(f_i(n)) = \operatorname{top}(f_{i_n}(n))$.

If t is the point of X with $t(n) = \operatorname{top}(f_{i_n}(n))$, then $t \in O \in \mathcal{O}$. So there is $k \in \omega$ and for each $n < k$ a clopen O_n in X_n such that $t(n) \in O_n$ and $(\prod_{n<k} O_n \times \prod_{n \geq k} X_n) \subset O$.

If $i \geq i_n$ for all $n < k$, $n < k$ implies $n < k_i$ and $\operatorname{top}(f_i(n)) = t(n)$. So regardless of $\operatorname{top}(f_i(n))$ for $k \leq n < k_i$, Case (1) holds for i. This contradicts the fact that Case (2) holds infinitely often.

References

1. R. Telgársky, *Total paracompactness and paracompact dispersed spaces*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **16** (1968), 567–572.

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