THE COMBINATORICS OF CERTAIN PRODUCTS

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Abstract. A combinatorial interpretation for the coefficients in the expansion of $\prod (1 + ux^j y^k)(1 - ux^j y^k)^{-1}$ is given.

1. Introduction. The coefficients $A(n; x, y)$, $B(n; x, y)$ and $C(n; x, y)$ defined by

\begin{align*}
\sum_{n \geq 0} \frac{A(n; x, y)u^n}{(x)_n(y)_n} &= \frac{1}{(u; x, y)}, \\
\sum_{n \geq 0} \frac{B(n; x, y)u^n}{(x)_n(y)_n} &= (-u; x, y), \\
\sum_{n \geq 0} \frac{C(n; x, y)u^n}{(x)_n(y)_n} &= \frac{(-u; x, y)}{(u; x, y)},
\end{align*}

where

\begin{align*}
(x)_n &= (1 - x)(1 - x^2) \cdots (1 - x^n), \\
(x)_0 &= 1,
\end{align*}

\begin{align*}
(u; x, y) &= \prod_{j, k \geq 0} (1 - ux^j y^k),
\end{align*}

have been considered by a number of mathematicians. Carlitz [2] was the first to demonstrate that these coefficients are polynomials in $x$ and $y$ with positive integral coefficients and provided closed formulas for calculating them. In a subsequent paper Roselle [6] gave combinatorial interpretations for $A(n; x, y)$ and $B(n; x, y)$. More recently, generalizations of (1.1) and (1.2) have appeared in [3, 4, 5] and [4], respectively.

A natural question, which was suggested to me by Dominique Foata, arises from Roselle's work: Is there a combinatorial interpretation for $C(n; x, y)$? This note provides one in terms of the already known interpretations of $A(n; x, y)$ and $B(n; x, y)$ and in terms of certain statistics defined on bicolored permutations.

2. Roselle's work. The polynomials $A(n; x, y)$ and $B(n; x, y)$ may be interpreted as the generating functions for various statistics defined on the group $G(n)$ consisting of permutations of $\{1, 2, \ldots, n\}$. To be precise, if the major index of a permutation $\sigma \in G(n)$, denoted $m(\sigma)$, is defined to be the sum of the elements in the set

\begin{align*}
\{ j : \sigma(j) > \sigma(j + 1), 1 \leq j \leq n-1 \}
\end{align*}

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and the $i$-major and co-$i$-major indices are defined, respectively, to be
\begin{align}
    i(\sigma) &= m(\sigma^{-1}), \\
    c(\sigma) &= \binom{n}{2} - i(\sigma),
\end{align}

then, as Roselle [6] and Foata [4] have demonstrated,
\begin{align}
    A(n; x, y) &= \sum_{\sigma \in G(n)} x^{m(\sigma)} y^{i(\sigma)}, \\
    B(n; x, y) &= \sum_{\sigma \in G(n)} x^{m(\sigma)} y^{c(\sigma)}.
\end{align}

3. Bicolored permutations. Since Carlitz [2] showed that $C(n; 1, 1) = 2^n \cdot n!$, it is clear that $G(n)$ will not provide a combinatorial structure for interpreting $C(n; x, y)$. For this reason, one is led to consider the set of bicolored permutations $BG(n)$, consisting of words $b = b(1)b(2) \cdots b(n)$ obtained by coloring each letter of some permutation $\sigma = \sigma(1)\sigma(2) \cdots \sigma(n) \in G(n)$ either red (indicated by underlining the letter) or blue (letter not underlined). For instance,
\begin{align}
    b = 3 \underline{2} 7 4 \underline{1} 6 5 \in BG(7).
\end{align}

Note that there are $2^n$ different colorings of each $\sigma \in G(n)$.

Now, possessing a combinatorial set with the correct cardinality, all that remains is to define some appropriate statistics on $BG(n)$ for which $C(n; x, y)$ is the generating function. To this end, let $W(n)$ be the set of words $w = w(1)w(2) \cdots w(n)$ of length $n$ with letters $w(i) \in \{0, 1\}$, and let $|w|$ denote the number of letters equal to 1 in $w$. The reduction of a permutation $\gamma$ of $\{a_1 < a_2 < \cdots < a_n\}$ is obtained by replacing $a_i$ in $\gamma$ by $i$ for $1 \leq i \leq n$. For instance, the reduction of the permutation $\gamma = 2416$ of $\{2 < 4 < 6\}$ is $2314$.

Each $b \in BG(n)$ is now assigned to a 4-tuple $(w, v, \theta, \alpha)$ where $w, v \in W(n)$ with $|w| = |v|$, $\theta \in G(|w|)$ and $\alpha \in G(n - |w|)$ according to the following rules:
\begin{align}
    w(i) &= \begin{cases} 
        1 & \text{if } b(i) \text{ is red}, \\
        0 & \text{otherwise},
    \end{cases} \\
    v(i) &= \begin{cases} 
        1 & \text{if } i \text{ is red in } b, \\
        0 & \text{otherwise},
    \end{cases} \\
    \theta &= \text{reduction of the red subpermutation of } b, \\
    \alpha &= \text{reduction of the blue subpermutation of } b.
\end{align}

Roughly speaking, $w$ indicates which positions of $b$ are red, $v$ which elements of $\{1, 2, \ldots, n\}$ are placed in the red positions, $\theta$ how the red letters are arranged in the red positions, and $\alpha$ how the blue letters are placed in the remaining positions. For instance, the bicolored permutation $b$ of (3.1) corresponds to the 4-tuple $(w, v, \theta, \alpha)$ where $w = 0 1 0 1 1 1 0, v = 1 1 0 1 0 1 0, \theta = 2314$ and $\alpha = 132$. Note that the map $b \rightarrow (w, v, \theta, \alpha)$ is a bijection between $BG(n)$ and the set of such 4-tuples.

Finally, for the bicolored permutation $b \rightarrow (w, v, \theta, \alpha)$ the appropriate statistics are defined by
\begin{align}
    M(b) &= m(\theta) + m(\alpha) + m(w), \\
    I(b) &= i(\theta) + c(\alpha) + m(v),
\end{align}
where \( m(w) \) and \( m(v) \) are defined in exactly the same way as the major index of a permutation.

4. The interpretation of \( C(n; x, y) \). It is now possible to show that
\[
C(n; x, y) = \sum_{b \in BG(n)} x^{M(b)} y^{l(b)}.
\]

First, as Carlitz [2] demonstrated, identities (1.1)–(1.3) imply
\[
C(n; x, y) = \sum_{k \geq 0} \binom{n}{k} x^k A(k; x, y) B(n - k; x, y),
\]
where the \( x \)-binomial coefficient is defined by
\[
\binom{n}{k}_x = \frac{(x)_n}{(x)_k (x)_{n-k}}.
\]

Second, the fact that (see [1, p.40])
\[
\sum x^{m(w)} = \binom{n}{k}_x,
\]
where the summation is over \( w \in W(n) \) with \(|w| = k\), along with (2.4), (2.5), (3.6) and (3.7) allow the calculation
\[
\sum_{b \in BG(n)} x^{M(b)} y^{l(b)} = \sum_{(w, v, \theta, \alpha)} x^{m(\theta) + m(\alpha) + m(w)} y^{l(\theta) + c(\alpha) + m(v)}
\]
\[
= \sum_{k \geq 0} \sum_{|w| = k} x^{m(w)} \sum_{|v| = k} y^{m(v)} \sum_{\theta \in G(k)} x^{m(\theta)} y^{l(\theta)} \sum_{\alpha \in G(n-k)} x^{m(\alpha)} y^{c(\alpha)}
\]
\[
= \sum_{k \geq 0} \binom{n}{k}_x \binom{n}{k}_y A(k; x, y) B(n - k; x, y).
\]

Identities (4.2) and (4.5) imply (4.1).

References

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