PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 89, Number 3, November 1983

CORRECTION TO "THE DIFFERENTIABILITY OF RIEMANN'S FUNCTION"

ARTHUR SMITH

Derivatives at other points. Suppose x is irrational, which, without essential loss, we take to be positive.

In the expansions leading to equations (10) and (11) of the above paper, put $x = x_n$, where $x_n = r_n/s_n$ and $r_n s_n \equiv 0 \pmod 2$. Put $h^2 = h_n^2 = 1/s_n^3$. Noticing we have an expansion in $h_n s_n$, we see the error term is $O(h_n^3 s_n^2)$, therefore, using Lemma 3, (10) may be written

$$\text{(i)} \qquad F(x_n \pm 1/s_n^3) = f(x_n) + A^{\pm}/s_n^2 + O(s_n^{-5/2}),$$

where $A^{\pm}$ is a number bounded as $r_n, s_n \to \infty$ and fixed when r_n and s_n are fixed modulo 4.

It is easy to prove that if x is irrational, $x_n \to x$ in such a way that $|x - x_n| = O(s_n^{-2})$, and f is differentiable at x, then $\lim_{n\to\infty} B_n^{\pm}$ exists with value 0.

As shown in the Lemma below, it is possible to find an infinite sequence of rationals $\{x_n\}$ with $x_n \to x$, $r_n s_n \equiv 0 \pmod 2$ and $|x - x_n| < 1/s_n^2$. Hence, it is possible to find an infinite sequence of rationals x_n all satisfying one of the further restraints, $s_n \equiv 1 \pmod 4$, $s_n \equiv 3 \pmod 4$, $r_n \equiv 1 \pmod 4$, $r_n \equiv 3 \pmod 4$. With the proper choice of sign (say $+$) we can then assume, by Lemma 3, that $A^+ \neq 0$ and fixed. From (i) this leads to the contradiction that $|B_n^+|$ is bounded away from 0, so f cannot be differentiable at x.

LEMMA. *If x is real and irrational there are infinitely many distinct rational numbers p/q such that $(p, q) = 1$, $pq \equiv 0 \pmod 2$ and $|x - p/q| < 1/q^2$.*

Without the constraint $pq \equiv 0 \pmod 2$ this is a standard theorem [**1**, p. 174]. Given the constraint we proceed as follows.

Suppose a/b, a'/b' are consecutive Farey fractions of order n and $x \in (a/b, a'/b')$.

If (i) $ab \equiv 0 \pmod 2$ and $a'b' \equiv 0 \pmod 2$, then standard methods apply to find p/q. Since $a'b - ab' = 1$, the other possibilities are (ii) $ab \equiv 1 \pmod 2$, $a'b' \equiv 0 \pmod 2$ or (iii) $ab \equiv 0 \pmod 2$, $a'b' \equiv 1 \pmod 2$.

In case (ii) let

$$I_k = \left(\frac{ka + a'}{kb + b'}, \frac{(k-1)a + a'}{(k-1)b + b'} \right), \qquad k = 1, 2, \ldots.$$

Received by the editors September 20, 1982.

AMS (MOS) subject classifications (1970). Primary 26A47; Secondary 10G05.

Key words and phrases. Riemann function, rational approximations to reals, nondifferentiability.

Note $x \in \bigcup_{k=1}^{\infty} I_k$ so there is a k such that $x \in I_k$. Furthermore, it is easy to show both endpoints of I_k satisfy constraints of type (i), and are consecutive Farey fractions of order $kb + b'$. Thus replacing a/b, a'/b' by the left- and right-hand endpoints of I_k, respectively, and n by $kb + b'$, we are in case (i).

Similarly, we can reduce case (iii) to case (i) by considering intervals

$$I_k = \left(\frac{a + (k-1)a'}{b + (k-1)b'}, \frac{a + ka'}{b + kb'} \right).$$

We may now proceed by standard methods to show infinitely many distinct primes exist with the required properties.

References

1. I. Niven and H. S. Zuckerman, *An introduction to the theory of numbers*, 4th ed., Wiley, New York, 1960.

2. Arthur Smith, *The differentiability of Riemann's function*, Proc. Amer. Math. Soc. **34** (1972), 463–468.

Department of Mathematics and Statistics, Carleton University, Ottawa, Ontario K1S 5B6, Canada