COHEN-MACAULAY ALGEBRAIC MONOIDS

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ABSTRACT. Let $E$ be an irreducible, normal, algebraic monoid with group of units $G = \text{Gl}_2(K), \text{Sl}_2(K) \times K^* \text{ or } \text{PGL}_2(K) \times K^*$. Then $E$ is a Cohen-Macaulay algebraic variety.

0. Introduction. An algebraic monoid $E$ is an algebraic variety together with an associative morphism $m: E \times E \to E$ and a two-sided unit $1 \in E$ for $m$.

Several authors [6, 8] are currently studying the theory of algebraic monoids. Prior to this, the first major geometric breakthrough [2] was published in 1972. In that work, Hochster proved that any normal $D$-monoid is a Cohen-Macaulay algebraic variety. The extent to which that result can be generalized to more general monoids is not known; this work is an optimistic addition.

Let $X$ be an irreducible algebraic variety. $X$ is Cohen-Macaulay if for all $x \in X$ there exists a regular sequence $(f_1, \ldots, f_m) \subseteq \mathfrak{c}_{x,X}$, which is a system of parameters.

Theorem. Suppose $E$ is an irreducible, normal, algebraic monoid with group of units $\text{Gl}_2(K), \text{PGL}_2(K) \times K^* \text{ or } \text{Sl}_2(K) \times K^*$. Then $E$ is a Cohen-Macaulay algebraic variety.

There are two main points to the proof.

1) In [11] it is proved that any finite, normal, separable, abelian covering of a smooth variety is Cohen-Macaulay. That result is generalized here.

2) Given any monoid $E$, as in the theorem, there exists a diagram

\[
\begin{array}{ccc}
E' & \xrightarrow{f} & \text{End}_K(K^2) \\
\downarrow g & & \downarrow \beta \\
E & \xrightarrow{\alpha} & E''
\end{array}
\]

such that all morphisms are finite and dominant and have $D$-group kernels. (1) above is then applied to the morphism $f$ to yield the result.

1. Notation and terminology. Let $K$ be an algebraically closed field. An algebraic monoid is an affine algebraic variety, defined over $K$, together with an associative morphism $m: E \times E \to E$ and a two-sided unit $1 \in E$ for $m$. Our general reference here is [6].
If $E$ is an algebraic monoid then $G(E) = \langle x \in E \mid x^{-1} \in E \rangle$ is an affine algebraic group and $G(E) \subseteq E$ is an open subset in the Zariski topology. $E^\circ$, the irreducible component of 1, is the unique maximal closed, irreducible submonoid of $E$.

An affine variety is completely determined by its $K$-algebra of regular functions. So, an affine, algebraic monoid $E$ is completely determined by morphisms $\Delta: K[E] \to K[E] \otimes K[E]$ and $\varepsilon: K[E] \to K$ such that

\[
(\Delta \otimes 1) \circ \Delta = (1 \otimes \Delta) \circ \Delta \quad \text{and} \quad (\eta \circ \varepsilon, 1) \circ \Delta = (1, \eta \circ \varepsilon) \circ \Delta = 1,
\]

where $\eta: K \to K[E]$ is the unit of the $K$-algebra structure of $K[E]$. $K[E]$ is thus a $K$-bigebra. The functor $E \to K[E]$ is an equivalence between the category of algebraic monoids and the category of finitely generated $K$-bigebra.

Let $Z$ be an irreducible, algebraic monoid. $Z$ is a $D$-monoid if $G(Z)$ is a torus. $X(Z) = \text{Hom}(Z, K)$ is the set of characters of $Z$. If $E$ is any irreducible algebraic monoid, a maximal $D$-submonoid $Z \subseteq E$ is the closure in $E$ of a maximal torus $T$ of $G(E)$. $Z$ is determined to within an isomorphism by the finitely generated, commutative monoid $X(Z)$.

An irreducible, algebraic monoid $E$ is reductive if $G(E)$ is a reductive algebraic group.

If $e = e^2 \in E$ let $G(e)$ denote the algebraic group of units of the algebraic monoid $eEe$.

$I(E) = \langle e \in E \mid e^2 = e \rangle$.

$E$ is regular if for all $x \in E$ there exists $g \in G(E)$ and $e \in I(E)$, such that $x = ge$.

For a survey of many of the known results on algebraic monoids the reader should consult [6]. A systematic treatment of some of the fundamental topics has been initiated in [8].

2. Finite $D$-group actions. To establish the main results of the paper, I have been led to consider certain morphisms $f: G \to H$ such that kernel($f$) is a (not necessarily reduced) finite $D$-group scheme. The purpose of this section is to generalize the results of P. Roberts [11] that can be applied to these morphisms $f$ above. Our reference for the theory of $D$-groups is [13, 2.2]. The subtlety arises only if the characteristic of the ground field divides the order of the group scheme.

**Definition.** Let $G$ be an affine group scheme over $K$. $G$ is a finite $D$-group if:

(i) $\dim_K K[G] < \infty$,

(ii) $X(G) = \{a \in K[G] \mid \Delta(a) = a \otimes a \}$

is a $K$-linear basis for $K[G]$, where $\Delta: K[G] \to K[G] \otimes K[G]$ is the comorphism induced from the multiplication morphism $m: G \times G \to G$.

**Note.** $X(G)$ is always a group. It is the group of characters of $G$. $K[G]$ is thus isomorphic to the group algebra of $X(G)$ over $K$.

2.1 **Lemma.** Let $X$ be an affine variety defined over $K$. Then there is a canonical one-to-one correspondence between actions of the $D$-group $G$ on $X$, and direct sum decompositions $K[X] = \bigoplus_{\alpha \in X(G)} K[X]_\alpha$ such that $K[X]_\alpha \cdot K[X]_\beta \subseteq K[X]_{\alpha + \beta}$ for all $\alpha, \beta \in X(G)$.
Proof. Let \( R = K[\mathcal{X}] \). Given \( R = \bigoplus R_\alpha \), define \( \mu : R \to R \otimes K[\mathcal{G}] \) by letting \( \mu(x) = x \otimes \alpha \) if \( x \in R_\alpha \). Plainly, this is the comorphism of an action \( G \times X \to X \).

Conversely, given \( \varphi : G \times X \to X \) we have \( \varphi^* : R \to R \otimes K[\mathcal{G}] \) which determines on \( R \) the structure of a \( K[\mathcal{G}] \)-comodule (i.e. \( \varphi^* \) is coassociative and \( (1 \otimes \varepsilon) \circ \varphi^* = 1 \), where \( \varepsilon \) is the augmentation on \( K[\mathcal{G}] \)). One checks, using these two facts, that if \( \varphi^*(a) = \sum_{\alpha \in X(\mathcal{G})} a_\alpha \otimes \alpha \) then \( a = \sum_{\alpha \in X(\mathcal{G})} a_\alpha \) and \( (a_\alpha)_\beta = a_\alpha \) if \( \alpha = \beta \), and 0 otherwise. Thus, \( R = \bigoplus_{\alpha \in X(\mathcal{G})} R_\alpha \) where \( R_\alpha = \langle \alpha \in R | a = a_\alpha \rangle \). Q.E.D.

The remainder of this section is devoted to the task of sharpening some known results (see [11]) about Cohen-Macaulay rings and finite \( D \)-group actions. I have assumed throughout that all rings are finitely generated \( K \)-algebras and that \( K \) is an algebraically closed field.

Let \( X \) be an affine variety defined over \( K \), and let \( G \) be a finite \( D \)-group scheme such that \( \varphi : G \times X \to X \) is an action of \( G \) on \( X \). For example, if \( X \) is an algebraic group and \( G \) is a closed finite \( D \)-subgroup scheme, then \( G \times X \to X \), \( (g,x) \mapsto g \cdot x \), is an action of \( G \) on \( X \). Let \( A \) be the coordinate ring of \( X \).

2.2 Lemma. Let \( X, A, \varphi \) be as above and let \( A_0 = \langle x \in A | \varphi^*(x) = x \otimes 1 \rangle \). Then:

(i) \( A_0 \) is a subalgebra of \( A \).

(ii) The inclusion \( A_0 \to A \) is an integral extension.

(iii) \( A_\alpha \) is an \( A_0 \)-module for all \( \alpha \in X(\mathcal{G}) \).

(iv) If \( A \) is a normal domain, so is \( A_0 \).

(v) \( A_0 \) is a finitely generated \( K \)-algebra.

Proof. (i)–(iii) are straightforward. For (iv) let \( K \) and \( L \) be the quotient fields of \( A_0 \) and \( A \), respectively. Since \( A \) is normal it suffices to prove that \( A_0 = A \cap K \). But \( A \cap K \) is an \( X(\mathcal{G}) \)-graded subspace of \( A \) and \( K = L_0 \). Thus, \( A \cap K = A \cap L_0 \subseteq A_0 \).

(v) follows directly from (ii) and [1, Chapter 5, 1.9]. Q.E.D.

2.3 Proposition. Suppose \( G \times X \to X \) is an action of the finite \( D \)-group \( G \) on the normal affine variety \( X \). If \( X/G = \text{Spec}(A_0) \) is smooth, then \( X \) is Cohen-Macaulay.

Proof. By assumption \( A \) is a normal domain and \( A_0 \to A \) is a regular subalgebra. By (2.2), \( A \) is a finite \( A_0 \)-module, and then by [1, Chapter 7, 4.8] it is actually reflexive. Since \( A \) splits up as \( \bigoplus_{\alpha \in X(\mathcal{G})} A_\alpha \), each summand \( A_\alpha \) is reflexive. Consider now any nonzero \( A_\alpha \), and choose \( 0 \neq x \) in it. Suppose \( \alpha \) has order \( n \) in \( X(\mathcal{G}) \). Then the map \( y \mapsto x^{n-1} \cdot y \) is injective and sends \( A_\alpha \) into \( A_0 \). The image is an ideal that is reflexive and hence is divisorial [1, Chapter 7, 4.2]. But by assumption \( A_0 \) is regular, and hence all its divisorial ideals are projective [1, Chapter 7, 4.7]. Thus, each \( A_\alpha \) is projective and so \( A \) is projective. In particular, \( A \) is flat over \( A_0 \). This implies that \( A \) is Cohen-Macaulay [12, Chapter 4, D]. Q.E.D.

The following well-known result is also needed in the proof of the main theorem (3.2).

2.4 Proposition. Suppose \( X \) is an irreducible, affine, Cohen-Macaulay variety and \( G \times X \to X \) is an action of the finite \( D \)-group on \( X \). Then \( X/G \) is Cohen-Macaulay.
Proof. Let $A = K[X]$ and $A_0 = K[X/G]$. $A$ is Cohen-Macaulay as an $A$-module and thus, as an $A_0$-module. But $A$ splits up as $A_0 \oplus \sum_{\alpha \in X(G) \setminus \{0\}} A_\alpha$. Thus, each summand, in particular $A_0$, is a Cohen-Macaulay $A_0$-module. Q.E.D.

3. The main results.

3.1 Proposition. Let $G$ be one of the groups $\text{SL}_2(K) \times K^*$, $\text{GL}_2(K)$ or $\text{PGL}_2(K) \times K^*$, and suppose $E$ is an irreducible, normal, algebraic monoid with 0 and unit group $G$. Then there exists a commutative diagram of algebraic monoids with $R = \text{End}_K(K^2)$:

$$
\begin{array}{ccc}
X & \xrightarrow{h} & R \\
\downarrow & & \downarrow \\
E & \rightarrow & Y
\end{array}
$$

Furthermore,

(i) all morphisms are finite and dominant,

(ii) the kernel of each morphism is a finite $D$-group.

The proof of 3.1 is recorded in [9]. The main point is to construct the desired morphisms $\nu$ and $h$, first on the level of maximal $D$-submonoids, and then globally using the “big cell” [9, §3] $U \subseteq E$.

3.2 Theorem. Let $E$ be an irreducible, normal, algebraic monoid with unit group $G = \text{SL}_2(K) \times K^*$, $\text{GL}_2(K)$ or $\text{PGL}_2(K) \times K^*$. Then $E$ is Cohen-Macaulay.

Proof. Case 1. $E$ has a 0.

We have, from 3.1, the following commutative diagram with $R = \text{End}_K(K^2)$:

$$
\begin{array}{ccc}
X & \rightarrow & R \\
\downarrow & & \downarrow \\
E & \rightarrow & Y
\end{array}
$$

All morphisms have finite $D$-group kernels and $R$ is a smooth variety. By 2.3, $X$ is Cohen-Macaulay and thus, by 2.4, $E$ is Cohen-Macaulay.

Case 2. $E$ does not have a 0.

We may assume, since groups are smooth, that $G \subseteq E$. By [3, Corollary 1.4], $E$ has an idempotent $e \neq 1$ and from [5, Corollary 2], it follows that $e \in \overline{T}$, the closure in $E$ of some maximal torus $T$ of $G$. Let $e_0 \in \overline{T}$ be the minimal idempotent. Then $e_0$ is fixed by the Weyl group $W$, so by [7, Theorem 2.3], $e_0$ is in the closure of the identity component $S$ of $ZG$, the center of $G$. Thus, $\dim(e_0T) > 0$, since otherwise $e_0T = \langle e_0 \rangle$ for all tori ($e_0$ is central and all maximal tori are conjugate), contradicting the fact that $E$ does not have a 0. Hence $I(\overline{T}) = \langle 1, e_0 \rangle$ [4, Theorem 3.7]. Let $G' = (G, G)$. As $E$ is regular [10], the map $m: G' \times \overline{S} \rightarrow E$, $m(x, y) = x \cdot y$, is onto, and since $G'$ is a simple algebraic group, $m$ is finite-to-one. Thus, $E = G \cup G \cdot e$; $\dim G = 4$, $\dim(G \cdot e) = \dim(G') = 3$. But since $E$ is normal, the singular locus has codimension larger than two. Hence, by homogeneity of $E \setminus G = G \cdot e$, $E$ is actually nonsingular in this case. Q.E.D.
References

9. ——. Classification of semisimple rank one monoids (to appear).
10. ——. Reductive monoids are von Neumann regular (to appear).

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