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COHEN-MACAULAY ALGEBRAIC MONOIDS

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ABSTRACT. Let E be an irreducible, normal, algebraic monoid with group of units $G = \mathrm{Gl}_2(K)$, $\mathrm{Sl}_2(K) \times K^*$ or $\mathrm{PGl}_2(K) \times K^*$. Then E is a Cohen-Macaulay algebraic variety.

0. Introduction. An *algebraic monoid E* is an algebraic variety together with an associative morphism $m\colon E \times E \to E$ and a two-sided unit $1 \in E$ for m.

Several authors [**6**, **8**] are currently studying the theory of algebraic monoids. Prior to this, the first major geometric breakthrough [**2**] was published in 1972. In that work, Hochster proved that any normal D-monoid is a Cohen-Macaulay algebraic variety. The extent to which that result can be generalized to more general monoids is not known; this work is an optimistic addition.

Let X be an irreducible algebraic variety. X is *Cohen-Macaulay* if for all $x \in X$ there exists a regular sequence $\{f_1, \ldots, f_m\} \subseteq \mathcal{O}_{x,X}$, which is a system of parameters.

THEOREM. *Suppose E is an irreducible, normal, algebraic monoid with group of units* $\mathrm{Gl}_2(K)$, $\mathrm{PGl}_2(K) \times K^*$ *or* $\mathrm{Sl}_2(K) \times K^*$. *Then E is a Cohen-Macaulay algebraic variety.*

There are two main points to the proof.

(1) In [**11**] it is proved that any finite, normal, separable, abelian covering of a smooth variety is Cohen-Macaulay. That result is generalized here.

(2) Given any monoid E, as in the theorem, there exists a diagram

$$\begin{array}{ccc} E' & \xrightarrow{f} & \mathrm{End}_K(K^2) \\ g\downarrow & & \downarrow\beta \\ E & \xrightarrow[\alpha]{} & E'' \end{array}$$

such that all morphisms are finite and dominant and have D-group kernels. (1) above is then applied to the morphism f to yield the result.

1. Notation and terminology. Let K be an algebraically closed field. An *algebraic monoid* is an affine algebraic variety, defined over K, together with an associative morphism $m\colon E \times E \to E$ and a two-sided unit $1 \in E$ for m. Our general reference here is [**6**].

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If E is an algebraic monoid then $G(E) = \{x \in E | x^{-1} \in E\}$ is an affine algebraic group and $G(E) \subseteq E$ is an open subset in the Zariski topology. E°, the irreducible component of 1, is the unique maximal closed, irreducible submonoid of E.

An affine variety is completely determined by its K-algebra of regular functions. So, an affine, algebraic monoid E is completely determined by morphisms Δ: $K[E] \to K[E] \otimes K[E]$ and ε: $K[E] \to K$ such that

$$(\Delta \otimes 1) \circ \Delta = (1 \otimes \Delta) \circ \Delta \quad \text{and} \quad (\eta \circ \varepsilon, 1) \circ \Delta = (1, \eta \circ \varepsilon) \circ \Delta = 1,$$

where η: $K \to K[E]$ is the unit of the K-algebra structure of $K[E]$. $K[E]$ is thus a *K-bigebra*. The functor $E \to K[E]$ is an equivalence between the category of algebraic monoids and the category of finitely generated K-bigebras.

Let Z be an irreducible, algebraic monoid. Z is a *D-monoid* if $G(Z)$ is a torus. $X(Z) = \operatorname{Hom}(Z, K)$ is the set of *characters* of Z. If E is any irreducible algebraic monoid, a *maximal D-submonoid* $Z \subseteq E$ is the closure in E of a maximal torus T of $G(E)$. Z is determined to within an isomorphism by the finitely generated, commutative monoid $X(Z)$.

An irreducible, algebraic monoid E is *reductive* if $G(E)$ is a reductive algebraic group.

If $e = e^2 \in E$ let $G(e)$ denote the algebraic group of units of the algebraic monoid eEe.

$I(E) = \{e \in E | e^2 = e\}$.

E is *regular* if for all $x \in E$ there exists $g \in G(E)$ and $e \in I(E)$, such that $x = ge$.

For a survey of many of the known results on algebraic monoids the reader should consult **[6]**. A systematic treatment of some of the fundamental topics has been initiated in **[8]**.

2. Finite D-group actions. To establish the main results of the paper, I have been led to consider certain morphisms f: $G \to H$ such that kernel(f) is a (not necessarily reduced) finite D-group scheme. The purpose of this section is to generalize the results of P. Roberts **[11]** that can be applied to these morphisms f above. Our reference for the theory of D-groups is **[13**, 2.2]. The subtlety arises only if the characteristic of the ground field divides the order of the group scheme.

DEFINITION. Let G be an affine group scheme over K. G is a *finite D-group* if:

(i) $\dim_K K[G] < \infty$,

(ii) $X(G) = \{a \in K[G] \| \Delta(a) = a \otimes a\}$

is a K-linear basis for $K[G]$, where Δ: $K[G] \to K[G] \otimes K[G]$ is the comorphism induced from the multiplication morphism m: $G \times G \to G$.

Note. $X(G)$ is always a group. It is the group of characters of G. $K[G]$ is thus isomorphic to the group algebra of $X(G)$ over K.

2.1 LEMMA. *Let X be an affine variety defined over K. Then there is a canonical one-to-one correspondence between actions of the D-group G on X, and direct sum decompositions $K[X] = \bigoplus_{\alpha \in X(G)} K[X]_\alpha$ such that $K[X]_\alpha \cdot K[X]_\beta \subseteq K[X]_{\alpha+\beta}$ for all $\alpha, \beta \in X(G)$.*

PROOF. Let $R = K[X]$. Given $R = \oplus R_\alpha$, define μ: $R \to R \otimes K[G]$ by letting $\mu(x) = x \otimes \alpha$ if $x \in R_\alpha$. Plainly, this is the comorphism of an action $G \times X \to X$.

Conversely, given φ: $G \times X \to X$ we have φ^*: $R \to R \otimes K[G]$ which determines on R the structure of a $K[G]$-comodule (i.e. φ^* is coassociative and $(1 \otimes \varepsilon) \circ \varphi^* = 1$, where ε is the augmentation on $K[G]$). One checks, using these two facts, that if $\varphi^*(a) = \sum_{\alpha \in X(G)} a_\alpha \otimes \alpha$ then $a = \sum_{\alpha \in X(G)} a_\alpha$ and $(a_\alpha)_\beta = a_\alpha$ if $\alpha = \beta$, and 0 otherwise. Thus, $R = \oplus_{\alpha \in X(G)} R_\alpha$ where $R_\alpha = \{a \in R | a = a_\alpha\}$. Q.E.D.

The remainder of this section is devoted to the task of sharpening some known results (see [**11**]) about Cohen-Macaulay rings and finite D-group actions. I have assumed throughout that all rings are finitely generated K-algebras and that K is an algebraically closed field.

Let X be an affine variety defined over K, and let G be a finite D-group scheme such that φ: $G \times X \to X$ is an action of G on X. For example, if X is an algebraic group and G is a closed finite D-subgroup scheme, then $G \times X \to X$, $(g, x) \to g \cdot x$, is an action of G on X. Let A be the coordinate ring of X.

2.2 LEMMA. *Let X, A, φ be as above and let $A_0 = \{x \in A | \varphi^*(x) = x \otimes 1\}$. Then*:

(i) *A_0 is a subalgebra of A.*

(ii) *The inclusion $A_0 \to A$ is an integral extension.*

(iii) *A_α is an A_0-module for all $\alpha \in X(G)$.*

(iv) *If A is a normal domain, so is A_0.*

(v) *A_0 is a finitely generated K-algebra.*

PROOF. (i)–(iii) are straightforward. For (iv) let K and L be the quotient fields of A_0 and A, respectively. Since A is normal it suffices to prove that $A_0 = A \cap K$. But $A \cap K$ is an $X(G)$-graded subspace of A and $K = L_0$. Thus, $A \cap K = A \cap L_0 \subseteq A_0$. (v) follows directly from (ii) and [**1**, Chapter 5, 1.9]. Q.E.D.

2.3 PROPOSITION. *Suppose $G \times X \to X$ is an action of the finite D-group G on the normal affine variety X. If $X/G =$ Spec(A_0) is smooth, then X is Cohen-Macaulay.*

PROOF. By assumption A is a normal domain and $A_0 \to A$ is a regular subalgebra. By (2.2), A is a finite A_0-module, and then by [**1**, Chapter 7, 4.8] it is actually reflexive. Since A splits up as $\oplus_{\alpha \in X(G)} A_\alpha$, each summand A_α is reflexive. Consider now any nonzero A_α, and choose $0 \neq x$ in it. Suppose α has order n in $X(G)$. Then the map $y \to x^{n-1} \cdot y$ is injective and sends A_α into A_0. The image is an ideal that is reflexive and hence is divisorial [**1**, Chapter 7, 4.2]. But by assumption A_0 is regular, and hence all its divisorial ideals are projective [**1**, Chapter 7, 4.7]. Thus, each A_α is projective and so A is projective. In particular, A is flat over A_0. This implies that A is Cohen-Macaulay [**12**, Chapter 4, D]. Q.E.D.

The following well-known result is also needed in the proof of the main theorem (3.2).

2.4 PROPOSITION. *Suppose X is an irreducible, affine, Cohen-Macaulay variety and $G \times X \to X$ is an action of the finite D-group on X. Then X/G is Cohen-Macaulay.*

PROOF. Let $A = K[X]$ and $A_0 = K[X/G]$. A is Cohen-Macaulay as an A-module and thus, as an A_0-module. But A splits up as $A_0 \oplus \sum_{\alpha \in X(G)\setminus\{0\}} A_\alpha$. Thus, each summand, in particular A_0, is a Cohen-Macaulay A_0-module. Q.E.D.

3. The main results.

3.1 PROPOSITION. *Let G be one of the groups* $\mathrm{Sl}_2(K) \times K^*$, $\mathrm{Gl}_2(K)$ *or* $\mathrm{PGl}_2(K) \times K^*$, *and suppose E is an irreducible, normal, algebraic monoid with* 0 *and unit group G. Then there exists a commutative diagram of algebraic monoids with* $R = \mathrm{End}_K(K^2)$:

$$\begin{array}{ccc} X & \xrightarrow{h} & R \\ {\scriptstyle v}\downarrow & & \downarrow \\ E & \rightarrow & Y \end{array}$$

Furthermore,

(i) *all morphisms are finite and dominant,*

(ii) *the kernel of each morphism is a finite D-group.*

The proof of 3.1 is recorded in [**9**]. The main point is to construct the desired morphisms v and h, first on the level of maximal D-submonoids, and then globally using the "big cell" [**9**, §3] $U \subseteq E$.

3.2 THEOREM. *Let E be an irreducible, normal, algebraic monoid with unit group* $G = \mathrm{Sl}_2(K) \times K^*$, $\mathrm{Gl}_2(K)$ *or* $\mathrm{PGl}_2(K) \times K^*$. *Then E is Cohen-Macaulay.*

PROOF. *Case* 1. E has a 0.

We have, from 3.1, the following commutative diagram with $R = \mathrm{End}_K(K^2)$:

$$\begin{array}{ccc} X & \rightarrow & R \\ \downarrow & & \downarrow \\ E & \rightarrow & Y \end{array}$$

All morphisms have finite D-group kernels and R is a smooth variety. By 2.3, X is Cohen-Macaulay and thus, by 2.4, E is Cohen-Macaulay.

Case 2. E does not have a 0.

We may assume, since groups are smooth, that $G \subsetneqq E$. By [**3**, Corollary 1.4], E has an idempotent $e \neq 1$ and from [**5**, Corollary 2], it follows that $e \in \overline{T}$, the closure in E of some maximal torus T of G. Let $e_0 \in \overline{T}$ be the minimal idempotent. Then e_0 is fixed by the Weyl group W, so by [**7**, Theorem 2.3], e_0 is in the closure of the identity component S of ZG, the center of G. Thus, $\dim(e_0T) > 0$, since otherwise $e_0T = \{e_0\}$ for all tori (e_0 is central and all maximal tori are conjugate), contradicting the fact that E does not have a 0. Hence $I(\overline{T}) = \{1, e_0\}$ [**4**, Theorem 3.7]. Let $G' = (G, G)$. As E is regular [**10**], the map m: $G' \times \overline{S} \to E$, $m(x,y) = x \cdot y$, is onto, and since G' is a simple algebraic group, m is finite-to-one. Thus, $E = G \cup G \cdot e$; $\dim G = 4$, $\dim(G \cdot e) = \dim(G') = 3$. But since E is normal, the singular locus has codimension larger than two. Hence, by homogeneity of $E \setminus G = G \cdot e$, E is actually nonsingular in this case. Q.E.D.

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