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# FINITELY BOOLEAN REPRESENTABLE VARIETIES

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ABSTRACT. This paper gives a short, elementary proof of a result of Burris and McKenzie [2] stating that each variety Boolean representable by a finite set of finite algebras is the join of an abelian and a discriminator variety. An example showing that the Boolean product operator $\Gamma^a$ is not idempotent is included as well.

**1. Introduction.** The result just mentioned was obtained as a corollary of the authors' description of locally finite decidable varieties with modular congruence lattices, and it was asked in Freese and McKenzie [3] whether there is a "reasonable" proof of this statement. We hope that the one given in the next section is such. In fact, we prove a bit more.

THEOREM. *Let $V$ be a variety generated by a finite set $\mathcal{K}$ of finite algebras with the property that the countable members of $V$ are in $\Gamma^a(\mathcal{K})$. Then $V$ is the join of an abelian and a discriminator variety* (*which are independent*).

We assume that the reader is familiar with the concept of modular commutator as well as that of Boolean product; these notions together with a complete background of the problem are found in Freese and McKenzie [3].

**2. The proof.** We fix a variety $V$ satisfying the conditions of the Theorem. By the results of McKenzie [5] (also mentioned in [3]), we may assume that $V$ is congruence permutable and each directly indecomposable algebra of $V$ is finite, and is either simple or abelian. Our reasoning is based on the following concept.

DEFINITION. *A subalgebra $\mathfrak{A}$ of an algebra $\mathfrak{B}$ is called very skew if $\mathfrak{A}$ is skew in each direct decomposition of $\mathfrak{B}$, that is, for nontrivial congruences $\theta, \psi$ of $\mathfrak{B}$ with $\theta \circ \psi = 1$, $\theta \cdot \psi = 0$ we have*

$$(\theta \restriction \mathfrak{A}) \circ (\psi \restriction \mathfrak{A}) < 1_{\mathfrak{A}}.$$

First, we show that a variety is of the desired type iff the powers of its neutral simple algebras have no large very skew subalgebras.

LEMMA 1. *Let $V$ be a finitely generated modular variety with the property that each directly indecomposable member of $V$ is either simple or abelian. Then $V$ is the join of an abelian and a discriminator variety iff for each neutral simple member $\mathfrak{B}_0$ of $V$ there exists a natural number $k$ such that the very skew subalgebras of the finite direct powers of $\mathfrak{B}_0$ admit at most $k$ elements.*

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Before proving this statement, we borrow an observation from [**4**] (where all modular varieties with complemented principal congruences are described) which will be useful later.

LEMMA 2. *Let the algebra* $\mathfrak{C}$ *(in a modular variety) be a subdirect product of some neutral simple algebras* $\mathfrak{C}_i$ $(i \in I)$, *and let* $\theta, \psi$ *be complements in the congruence lattice of* $\mathfrak{C}$. *Then for some subsets* $A$ *and* $B = I - A$ *of* $I$, *the congruences* $\theta$ *and* $\psi$ *are just the kernels of the projections to* $A$ *and* $B$, *respectively.*

SKETCH OF PROOF. Let $\phi_i$ be the kernel of the projection to $\mathfrak{C}_i$. Then as $\mathfrak{C}_i$ is neutral and simple, each $\phi_i$ is either over $\theta$ or over $\psi$. Now the choice $A = \{i \mid \phi_i \geqslant \theta\}$ works by modularity.

Let us now prove Lemma 1. The 'only if' part is clear by standard arguments (see [**6**]), and by [**1**, Corollary 9.9] it suffices to show that if $\mathfrak{B}_0$ is a neutral simple algebra of $V$, and $\mathfrak{A}_0$ is a nonsingleton subalgebra of $\mathfrak{B}_0$, then $\mathfrak{A}_0$ is neutral and simple.

First, let $\phi$ be a nontrivial congruence of $\mathfrak{A}_0$, $\mathfrak{B} = {}^n(\mathfrak{B}_0)$, and set

$$\mathfrak{A} = \left\{\mathbf{b} \in {}^n(\mathfrak{A}_0) \mid b_i \phi b_j \ (i, j \in n)\right\} \leqslant \mathfrak{B}$$

(this construction is standard in commutator theory). Now $\mathfrak{A}$ is very skew if $\phi \neq 1$ (since the direct decompositions of $\mathfrak{B}$ are the obvious ones by Lemma 2), and if $\phi \neq 0$, then $\mathfrak{A}$ has at least $2^n$ elements, which is a contradiction.

Secondly, suppose the $\mathfrak{A}_0$ is abelian, set $\mathfrak{B} = {}^{2n}(\mathfrak{B}_0)$ and

$$\mathfrak{A} = \left\{\mathbf{b} \in {}^{2n}(\mathfrak{A}_0) \mid b_0 + \cdots + b_{n-1} = b_n + \cdots + b_{2n-1}\right\},$$

where $+$ is an abelian group addition on $\mathfrak{A}_0$ compatible with the fundamental operations. This $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}$ of cardinality $|\mathfrak{A}_0|^{2n-1}$. Furthermore, the images of $\mathfrak{A}$ and ${}^{2n}(\mathfrak{A}_0)$ are the same in each proper factor of $\mathfrak{B}$, thus $\mathfrak{A}$ is again very skew.

The appropriate sensitive tool for investigating very skew subalgebras seems to be the following.

DEFINITION. *Let* $\mathfrak{A} \leqslant \mathfrak{B}$ *be algebras. Set*

$$\mathfrak{B}_{[\mathfrak{A}]} = \{u \in {}^{\omega}\mathfrak{B} \mid \exists a \in \mathfrak{A} \colon \{i \mid u_i \neq a\} \text{ is finite}\}.$$

*We denote this element a by* $\hat{u}$, *and for* $b \in \mathfrak{B}$ *let* $\bar{b}\colon \omega \to \mathfrak{B}$ *be the constant b mapping.*

It will turn out that $\mathfrak{A}$ "splits nicely" if we make direct decompositions of $\mathfrak{B}$ with the aid of a Boolean product representation of $\mathfrak{B}_{[\mathfrak{A}]}$. More precisely, we show that if $\mathfrak{B} = {}^n(\mathfrak{B}_0)$ for some neutral simple (finite) $\mathfrak{B}_0$, and $\mathfrak{A} \leqslant \mathfrak{B}$ is very skew, then $\mathfrak{B}_{[\mathfrak{A}]}$ has only trivial direct decompositions, and hence if $\mathfrak{B}_{[\mathfrak{A}]}$ is in $\Gamma^a(\mathfrak{K})$, then some element of $\mathfrak{K}$ majorates $\mathfrak{A}$ in power. Let us fix the algebras $\mathfrak{B}_0$, $\mathfrak{A}$, $\mathfrak{B}$ as in the previous sentence.

LEMMA 3. *Suppose that* $\mathfrak{B}_{[\mathfrak{A}]} \cong \mathfrak{C}_1 \times \mathfrak{C}_2$. *Then either* $\mathfrak{C}_1$ *or* $\mathfrak{C}_2$ *is finite. If* $a_1, a_2$ *are different elements of* $\mathfrak{A}$ *then the images of* $\bar{a}_1$ *and* $\bar{a}_2$ *in the "cofinite" component of this direct decomposition are also different.*

PROOF. We have a subdirect decomposition

$$\mathfrak{B}_{[\mathfrak{A}]} \leqslant \prod_{n \times \omega} \mathfrak{B}_0.$$

Let $\theta$, $\psi$ be the congruences corresponding to the decomposition $\mathfrak{B}_{[\mathfrak{A}]} \cong \mathfrak{C}_1 \times \mathfrak{C}_2$, and $A, B \subseteq n \times \omega$ be the subsets given by Lemma 2. It clearly suffices to show that either $A$ or $B$ is finite.

For each $i \in \omega$ we get a direct decomposition of $\mathfrak{B}$ from that of $\mathfrak{B}_{[\mathfrak{A}]}$: it is determined by the subsets

$$A_i = \{j \in n \mid (j, i) \in A\} \quad \text{and} \quad B_i = \{j \in n \mid (j, i) \in B\}$$

of $n$. We prove that disregarding finitely many indices $i$, this decomposition is trivial. Indeed, otherwise, there is an infinite $I \subseteq \omega$ such that $A_i$ and $B_i$ are the same subsets, say $A'$ and $B'$, of $n$, respectively, for $i \in I$; and $A', B' \neq \varnothing$. Let $\theta'$, $\psi'$ denote the congruences of $\mathfrak{B}$ corresponding to its direct decomposition determined by $A'$ and $B'$. As $\mathfrak{A}$ is very skew, there exist elements $a_1, a_2$ of $\mathfrak{A}$ such that

$$(a_1, a_2) \notin (\theta' \restriction \mathfrak{A}) \circ (\psi' \restriction \mathfrak{A}).$$

On the other hand, $(\bar{a}_1, \bar{a}_2) \in \theta \circ \psi$, say $\bar{a}_1 \theta u \psi \bar{a}_2$, and as $I$ is infinite, we clearly have $a_1 \theta \hat{u} \psi' a_2$, which is a contradiction.

Suppose now that $A_i = \varnothing$, as well as $B_i = \varnothing$, hold infinitely many times. Choose arbitrary elements $a_1 \neq a_2$ from $\mathfrak{A}$. Then with some $\bar{a}_1 \theta u \psi \bar{a}_2$ we clearly have $a_1 = \hat{u} = a_2$, which is a contradiction. Thus, either $A$ or $B$ is finite, as desired.

The proof of the Theorem will be complete by showing

LEMMA 4. *If $\mathfrak{B}_{[\mathfrak{A}]} \in \Gamma^a(\mathcal{K})$, then there exists a $\mathfrak{K} \in \mathcal{K}$ such that $|\mathfrak{A}| \leqslant |\mathfrak{K}|$.*

PROOF. We have

$$\mathfrak{B}_{[\mathfrak{A}]} \leqslant_{bp} \prod_{i \in I} \mathfrak{K}_i.$$

If for some $i \in I$, the $i$th components of the elements $\bar{a}$ ($a \in \mathfrak{A}$) are all different, then clearly $|\mathfrak{A}| \leqslant |\mathfrak{K}_i|$. Otherwise, we have (with $[\![x = y]\!]$ being the equalizer of $x$ and $y$)

$$\bigcup_{a_1 \neq a_2 \in \mathfrak{A}} [\![\bar{a}_1 = \bar{a}_2]\!] = I.$$

Thus we obtain a partition of $I$ into the clopen sets $A_1, \ldots, A_s$ with the property that each $A_i$ is covered by some $[\![\bar{a}_1 = \bar{a}_2]\!]$. This partition defines a direct decomposition of $\mathfrak{B}_{[\mathfrak{A}]}$ which does not satisfy Lemma 3.

**3. $\Gamma^0$ is not idempotent.** If all the elements of $\mathcal{K}$ are either affine or simple, then we have a straightforward proof to the Theorem by constructing a term in $F_V(\omega)$ which is the discriminator in each maximal neutral simple algebra of $V$; and each variety representable by any $\mathcal{K}$ can be represented by such a $\mathcal{K}$ by the Theorem and **[6]**. However, our previous argument shows that we cannot in general assume that $\mathcal{K}$ is so nice. Indeed, let $\mathfrak{B}_0$ be the alternating group on five letters, $\mathfrak{A}_0$ a two-element subgroup of $\mathfrak{B}_0$ and let $\mathfrak{B}, \mathfrak{A}$ be as in the "abelian" construction of the proof of

Lemma 1 for some $n$ with the property that $|\mathfrak{A}| = 2^{2^{n-1}} > |\mathfrak{B}_0|$. Then, by Lemma 4, we have

$$\mathfrak{B}_{[\mathfrak{A}]} \in \Gamma^a(\mathfrak{A}, \mathfrak{B}) - \Gamma^a(\mathfrak{A}_0, \mathfrak{B}_0),$$

and since clearly $\mathfrak{A}, \mathfrak{B} \in \Gamma^a(\mathfrak{A}_0, \mathfrak{B}_0)$, this example shows also that the operator $\Gamma^a$ is not idempotent.

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