Cyclic Algebras of Small Exponent

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Abstract. We prove that every cyclic algebra of exponent \( n \) and degree \( mn \) over a field containing a primitive \( nth \) root of unity is similar to a tensor product of at most \( m \) symbols of degree \( n \).

1. Introduction. Let \( F \) be a field containing a primitive \( nth \) root of unity \( \omega \). A central simple \( F \)-algebra of degree \( n \) (i.e. of dimension \( n^2 \)) is called a symbol if it is generated by two elements \( x, y \) subject to the relations \( x^n \in F^*, y^n \in F^* \) and \( yx = \omega xy \) (compare [3, §15]). Merkurjev and Suslin [2] have recently proved that every finite-dimensional central simple \( F \)-algebra of exponent \( n \) (i.e. whose similarity class has order \( n \) in the Brauer group \( Br(F) \)) is similar to a tensor product of symbols of degree \( n \).

The aim of this note is to give a simple proof of this theorem for cyclic algebras, i.e. for central simple algebras which contain a cyclic extension of the center as a maximal commutative subalgebra.

Theorem. Let \( F \) be a field containing a primitive \( nth \) root of unity. Every cyclic \( F \)-algebra of exponent \( n \) and degree \( mn \) is similar to a tensor product of at most \( m \) symbols of degree \( n \).

If \( K \) is an extension of a field \( F \), we denote by \( Br(K/F) \) the kernel of the natural map from \( Br(F) \) to \( Br(K) \) and by \( Br_n(K/F) \) the subgroup of \( Br(K/F) \) which is killed by \( n \).

2. Lemma. Let \( K/F \) be a cyclic field extension and let \( L \) be an intermediate field. Let \( n = [K:L] \). Then, the image of the corestriction map

\[
\text{Cor}_{L/F} : Br(K/L) \to Br(K/F)
\]

is \( Br_n(K/F) \).

Proof. Let \( G \) be the Galois group of \( K \) over \( F \) and let \( \chi \) be a generator of the group \( \text{Hom}(G, \mathbb{Q}/\mathbb{Z}) \) of characters of \( G \). By [6, Corollary 2, p. 211], every element of \( Br(K/F) \) is of the form \( (\chi, a) \) for some \( a \in F^* \). (If we denote by \( \sigma \) the generator of \( G \) such that \( \chi(\sigma) = [K:F]^{-1} (\text{mod} \mathbb{Z}) \), then \( (\chi, a) \) is the similarity class of the cyclic algebra \( (K, \sigma, a) \), with the notations of [1, p. 74].) If \( (\chi, a) \) is killed by \( n \), then \( (n\chi, a) = 0 \).

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Since the kernel of $\chi$ is the Galois group of $K$ over $L$, we then have $a = N_{L/F}(t)$ for some $t \in L^*$, by [6, Corollary 1, p. 211], whence $(\chi, a) = \text{Cor}_{L/F}(\text{Res}_{L/F} \chi, t)$, by the “projection formula” (see [6, p. 212 or 7, Proposition 4.3.7]). This proves that the image of $\text{Br}(K/L)$ by the corestriction map contains $\text{Br}_n(K/F)$. The converse is clear, since the exponent of $\text{Br}(K/L)$ divides $[K : L] = n$. Q.E.D.

3. Proof of the Theorem. Let $K$ be a cyclic extension of $F$, of rank $mn$. The Lemma shows that every element in $\text{Br}_n(K/F)$ is the corestriction of some element in $\text{Br}(K/L)$, where $L$ is the (unique) extension of $F$ of codimension $n$ in $K$. Since $L$ contains a primitive root of unity, every element in $\text{Br}(K/L)$ is the similarity class of a symbol of degree $n$ and, since $[L : F] = m$, the corestriction of any symbol of degree $n$ over $L$ is similar to the tensor product of at most $m$ symbols of degree $n$ over $F$, by a theorem of Rosset and Tate [4, §3, Corollary 1]. Q.E.D.

4. Remarks. (1) If $n$ is a product of relatively prime integers $n = n_1 \cdots n_r$, then, by [1, Theorem 7.20], every cyclic algebra $A$ of exponent $n$ is isomorphic to a tensor product $A \cong A_1 \otimes \cdots \otimes A_r$, where $A_i$ is a cyclic algebra of exponent $n_i$ for $i = 1, \ldots, r$. The Theorem above can thus be applied separately to $A_1, \ldots, A_r$, this yields a better bound for the number of factors in a decomposition of $A$ as a tensor product of symbols (up to similarity). If $n$ is a power of a prime integer, it is not known whether the bound is the best possible. (It is obviously so for $m = 1$ or 2.)

(2) For $n = 2$, the Theorem above has also been proved by Rowen [5, Theorem 3.7], under the extra hypothesis that $F$ contains a primitive $2m$th root of unity. His techniques are different and do not yield a bound on the number of symbols.

REFERENCES


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