PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 89, Number 4, December 1983

PHRAGMÉN-LINDELÖF THEOREM IN A COHOMOLOGICAL FORM

CHING-HER LIN[1]

ABSTRACT. The main result of this paper is as follows. Given functions $\phi_1(\varepsilon),\dots,\phi_\nu(\varepsilon)$ which are holomorphic in sectors $S_1,\dots,S_\nu$, respectively, where $S_1 \cup \cdots \cup S_\nu = \{\varepsilon: |\arg \varepsilon| < \pi/2\alpha,\ 0 < |\varepsilon| < \rho\}$ for $\alpha > 1$, $\rho > 0$, set $\phi_{jk} = \phi_j - \phi_k$ if $S_j \cap S_k \neq \varnothing$. Then $\{\phi_{jk}\}$ satisfy cocycle conditions $\phi_{jk} + \phi_{kl} = \phi_{jl}$ whenever $S_j \cap S_k \cap S_l \neq \varnothing$. In addition to the conditions $|\phi_1| < M_0$ and $|\phi_\nu| < M_0$ on the two rays of the boundary (i.e. $\arg \varepsilon = \pi/2\alpha$), and $|\phi_j(\varepsilon)| \leqslant A \exp(c/|\varepsilon|)$ in S_j for some positive numbers A and c, $j = 1,2,\dots,\nu$, if the $\{\phi_j\}$ satisfy the conditions $|\phi_{jk}| < M_0$ on $S_j \cap S_k$ ($\neq \varnothing$), then we get $|\phi_j| < M$ on S_j, $j = 1,2,\dots,\nu$. (From the cohomological point of view, we can get global results for ϕ_j, once the local data on cocycles is known.)

1. Introduction. Let $\Omega = \{\varepsilon: -\pi/2\alpha < \arg \varepsilon < \pi/2\alpha,\ 0 < |\varepsilon| < \rho\}$ be a sector in the right half complex ε-plane, where $\alpha > 1$ and ρ is a positive number. Let f be a complex valued function which is continuous on $\Omega^* = \{\varepsilon: -\pi/2\alpha \leqslant \arg \varepsilon \leqslant \pi/2\alpha,\ 0 < |\varepsilon| \leqslant \rho\}$, holomorphic in Ω, and there are positive constants M and c such that $|f(\varepsilon)| \leqslant M \exp(c/|\varepsilon|)$ for all $\varepsilon \in \Omega$. Assume, furthermore, that $|f(\varepsilon)| \leqslant M$ for all ε on the boundary of Ω. Then, the Phragmén-Lindelöf theorem states that $|f(\varepsilon)| \leqslant M$ for all ε in Ω (see [**2**, p. 282]). In this paper, we shall generalize this theorem in a cohomological form; i.e. given functions $\phi_1(\varepsilon),\dots,\phi_\nu(\varepsilon)$ which are holomorphic in sectors $S_1,\dots,S_\nu$, respectively, $\Omega = S_1 \cup \cdots \cup S_\nu$, set $\phi_{jk} = \phi_j - \phi_k$ if $S_j \cap S_k \neq \varnothing$. Then $\{\phi_{jk}\}$ satisfy cocycle conditions $\phi_{jk} + \phi_{kl} = \phi_{jl}$ whenever $S_j \cap S_k \cap S_l \neq \varnothing$. With this property, our theorem can be stated in the following way: If the $\{\phi_j\}$ satisfy the conditions $|\phi_{jk}| \leqslant M_0$ on $S_j \cap S_k$ ($\neq \varnothing$) in addition to the conditions $|\phi_1| \leqslant M_0$ and $|\phi_\nu| \leqslant M_0$ on the two rays of the boundary (i.e. $\arg \varepsilon = \pm\pi/2\alpha$), then we get $|\phi_j| \leqslant M$ on S_j, $j = 1,\dots,\nu$ (cf. Theorem 1). From the cohomological point of view, we can get global results for ϕ_j once the local data on cocycles is known. In our theorem, we have chosen those covering sectors $S_1,\dots,S_\nu$ in a nice situation in which $S_j \cap S_k \neq \varnothing$ only for $k = j-1$ or $j+1$, $j = 2,\dots,\nu-1$ (cf. (2.1)). The following example will give a prototype of our result.

Let us consider two sectors $S_j = \{\varepsilon: a_j < \arg \varepsilon < b_j,\ 0 < |\varepsilon| < \rho\}$ ($j = 1,2$) where $-\pi/2\alpha = a_1 < a_2 < b_1 < b_2 = \pi/2\alpha$. Note that $S_1 \cup S_2 = \{\varepsilon: -\pi/2\alpha < \arg \varepsilon < \pi/2\alpha,\ 0 < |\varepsilon| < \rho\}$ and $S_1 \cap S_2 \neq \varnothing$ (cf. Figure 1).

Received by the editors December 9, 1982.
1980 *Mathematics Subject Classification*. Primary 30A10; Secondary 34E20.
Key words and phrases. Asymptotic theory, inequalities in the complex domain.
[1]This work was partially sponsored by the National Science Foundation under grant MCS 79-01998. This work was also a part of the author's dissertation for a Ph.D of Mathematics at the University of Minnesota, written under the supervision of Professor Sibuya.

0002-9939/83 $1.00 + $.25 per page

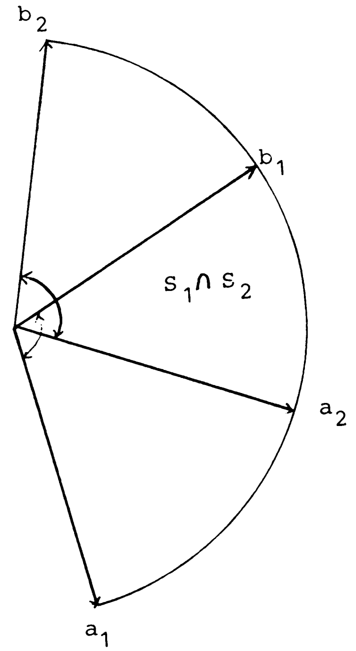

FIGURE 1

Let ϕ_j be functions of ε which are, respectively,

(1) holomorphic in S_j,

(2) continuous on $S_j^* = \{\varepsilon\colon a_j \leqslant \arg \varepsilon \leqslant b_j,\ 0 < |\varepsilon| \leqslant \rho\}$, and

(3) $|\phi_j(\varepsilon)| \leqslant A \exp(c/|\varepsilon|)$ in S_j for some positive numbers c and A.

Assume that $|\phi_2(\varepsilon) - \phi_1(\varepsilon)| \leqslant M_0$ in $S_1 \cap S_2$, $|\phi_1(\varepsilon)| \leqslant M_0$ on the line segment $\arg \varepsilon = -\pi/2\alpha$, $0 < |\varepsilon| < \rho$, and $|\phi_2(\varepsilon)| \leqslant M_0$ on the line segment $\arg \varepsilon = \pi/2\alpha$, $0 < |\varepsilon| < \rho$, for some M_0. Then we claim that $|\phi_j(\varepsilon)| \leqslant M$ in S_j ($j = 1, 2$), respectively, for some positive number M (cf. Theorem 1).

The difficulty in proving this result is due to the fact that ϕ_1 and ϕ_2 are two different functions in $S_1 \cap S_2$, and hence a straightforward application of the Phragmén-Lindelöf theorem does not work.

The proof of this result, may be obtained by defining an auxiliary function (as in the proof of the Phragmén-Lindelöf theorem) together with Theorem 2 (cf. §2). Previously, Y. Sibuya [3] proved a result similar to Theorem 2, in the case when $S_1 \cup S_2 \cup \cdots \cup S_\nu$ is a disk. However, he did not show that M was actually independent of c_1 as c_1 tends to zero. In the proof of Theorem 1, we shall let c_1 of Theorem 2 tend to zero (as in the proof of the Phragmén-Lindelöf theorem). Since M in Theorem 2 is independent of c_1 as c_1 tends to zero, such a process will work. Notice also that if $\bigcup_{j=1}^{\nu} S_j$ is a disc, there are no boundary rays. In this paper, $\bigcup_{j=1}^{\nu} S_j$ being a sector, the analysis near the boundary rays becomes another additional difficulty (cf. Proof of Theorem 2 in §3).

Given functions $\delta_1(\varepsilon), \ldots, \delta_\nu(\varepsilon)$ which are holomorphic in sectors $S_1, \ldots, S_\nu$, respectively, $S_1 \cup \cdots \cup S_\nu = \{\varepsilon\colon |\arg \varepsilon| < \pi/2\alpha,\ 0 < |\varepsilon| < \rho\}$, satisfying certain reasonable assumptions and having relatively poor estimates on the two rays of the boundary (i.e. $\arg \varepsilon = \pm\pi/2\alpha$) which are assumed to be close to the imaginary axis, we can substantially improve such estimates along the real axis utilizing Theorem 3 (cf. Theorem 3 in §2). The usefulness of our results is due to the fact that, in general, it is very difficult to get a global result all at once. In some cases where the Phragmén-Lindelöf theorem cannot be applied directly, we still can obtain a global result by putting suitable local results together through an application of our theorem (cf. e.g. [**1** and **3**]).

2. The statement of theorems.

THEOREM 1. *Let* $S_j = \{\varepsilon\colon a_j < \arg \varepsilon < b_j,\ 0 < |\varepsilon| < \rho\}$ $(j = 1,\ldots,\nu)$ *be sectors in the right half complex* ε*-plane where* $\rho > 0$:

(2.1)

$$-\pi/2\alpha = a_1 < a_2 < b_1 < a_3 < b_2 < a_4 < b_3 < \cdots < a_\nu < b_{\nu-1} < b_\nu = \pi/2\alpha, \quad \alpha > 1.$$

Let $\phi_1(\varepsilon),\ldots,\phi_\nu(\varepsilon)$ *be functions of* ε. *Assume that*

(1) $\phi_j(\varepsilon)$ *is holomorphic in* S_j *and continuous on* $S_j^* = \{\varepsilon\colon a_j \leq \arg \varepsilon \leq b_j,\ 0 < |\varepsilon| \leq \rho\}$,

(2) $|\phi_j(\varepsilon)| \leq A \exp(c/|\varepsilon|)$ *in* S_j, *for some positive numbers* A *and* c,

(3) $|\phi_{j+1}(\varepsilon) - \phi_j(\varepsilon)| < M_0$ *in* $S_j \cap S_{j+1}$, $|\phi_1(\varepsilon)| \leq M_0$ *on the line segment* $\arg \varepsilon = -\pi/2\alpha$, $0 < |\varepsilon| < \rho$, *and* $|\phi_\nu(\varepsilon)| \leq M_0$ *on the line segment* $\arg \varepsilon = \pi/2\alpha$, $0 < |\varepsilon| < \rho$, *for some positive number* M_0. *Then, there exists a positive number* M *such that* $|\phi_j(\varepsilon)| \leq M$ *in* S_j, $j = 1,\ldots,\nu$.

REMARK. The inequalities (2.1) mean that $S_j \cap S_k \neq \varnothing$ only for $k = j - 1$ or $j + 1$, $j = 2, 3,\ldots,\nu - 1$.

THEOREM 2. *Let* $S_j = \{\varepsilon\colon a_j < \arg \varepsilon < b_j,\ 0 < |\varepsilon| < \rho\}$ $(j = 1,2,\ldots,\nu)$ *be sectors in the complex* ε*-plane, where* $\rho > 0$:

$$-\pi < a_1 < a_2 < b_1 < a_3 < b_2 < a_4 < b_3 < \cdots < a_\nu < b_{\nu-1} < b_\nu < \pi.$$

Let $\delta_1(\varepsilon),\ldots,\delta_\nu(\varepsilon)$ *be functions of* ε. *Assume that*

(1) $\delta_j(\varepsilon)$ *is holomorphic in* S_j, *continuous on* S_j^*,

(2) $\delta_j(\varepsilon)$ *is asymptotically zero as* ε *tends to zero in* S_j, *i.e.* $|\delta_j(\varepsilon)| \leq K_N |\varepsilon|^N$ $(N = 0, 1, 2,\ldots)$ *in* S_j *for some positive numbers* K_N,

(3) $|\delta_{j+1}(\varepsilon) - \delta_j(\varepsilon)| \leq M_0 \exp(-c_1/|\varepsilon|^\lambda)$ *in* $S_j \cap S_{j+1}$, $|\delta_1(\varepsilon)| \leq M_0 \exp(-c_1/|\varepsilon|^\lambda)$ *on the line segment* $\arg \varepsilon = a_1$, $0 < |\varepsilon| < \rho$, *and* $|\delta_\nu(\varepsilon)| \leq M_0 \exp(-c_1/|\varepsilon|^\lambda)$ *on the line segment* $\arg \varepsilon = b_\nu$, $0 < |\varepsilon| < \rho$, *for some positive numbers* c_1, M_0 *and* λ. *Then, there exists a positive number* M *which is independent of* c_1 *as* c_1 *tends to zero such that*

$$|\delta_j(\varepsilon)| \leq M \exp\left(-c_1/|\varepsilon|^\lambda\right) \quad in\ S_j, j = 1, 2,\ldots,\nu.$$

THEOREM 3. *Let* $S_j = \{\varepsilon\colon a_j < \arg \varepsilon < b_j,\ 0 < |\varepsilon| < \rho\}$ *be sectors in the right half complex* ε*-plane, where* $\rho > 0$:

$$-\pi/2\alpha = a_1 < a_2 < b_1 < a_3 < b_2 < a_4 < b_3 < \cdots < a_\nu < b_{\nu-1} < b_\nu = \pi/2\alpha, \quad \alpha > 1.$$

Let $\delta_1(\varepsilon), \dots, \delta_\nu(\varepsilon)$ *be functions of* ε. *Assume that*

(1) $\delta_j(\varepsilon)$ *is holomorphic in* S_j, *continuous on* S_j^*,

(2) $\delta_j(\varepsilon)$ *is asymptotically zero, as* ε *tends to zero in* S_j,

(3) $|\delta_{j+1}(\varepsilon) - \delta_j(\varepsilon)| \leq M_0 \exp\{-\mu \operatorname{Re}(1/\varepsilon)\}$ *in* $S_j \cap S_{j+1}$,

$$|\delta_1(\varepsilon)| \leq M_0 \exp\{-\mu \operatorname{Re}(1/\varepsilon)\}$$

on the line segment $\arg \varepsilon = -\pi/2\alpha$, $0 < |\varepsilon| < \rho$, *and*

$$|\delta_\nu(\varepsilon)| \leq M_0 \exp\{-\mu \operatorname{Re}(1/\varepsilon)\}$$

on the line segment $\arg \varepsilon = \pi/2\alpha$, $0 < |\varepsilon| < \rho$, *for some positive numbers* μ *and* M_0. *Then, there exists a positive number* M *such that* $|\delta_j(\varepsilon)| \leq M \exp\{-\mu \operatorname{Re}(1/\varepsilon)\}$ *in* S_j, $j = 1, 2, \dots, \nu$.

3. Proof of the theorems.

(1) Proof of Theorem 2. We denote by V_j the intersections $S_j \cap S_{j+1}$, $j = 1, 2, \dots, \nu - 1$, respectively, and consider an open sector

$$S = \{\varepsilon\colon a_1 < \arg \varepsilon < b_\nu,\ 0 < |\varepsilon| < \rho_0\}, \quad \text{where } 0 < \rho_0 < \rho. \tag{3.1}$$

We choose $\nu - 1$ line segments $l_1, l_2, \dots, l_{\nu-1}$ such that $l_j \subset V_j$, i.e. l_j: $\varepsilon = te^{i\alpha_j}$ $(0 < t < \rho_0$, for some α_j in $(a_{j+1}, b_j))$. These $\nu - 1$ line segments divide the open sector S (cf. (3.1)) into ν open sectors $\hat{S}_1, \hat{S}_2, \dots, \hat{S}_\nu$ (cf. Figure 2).

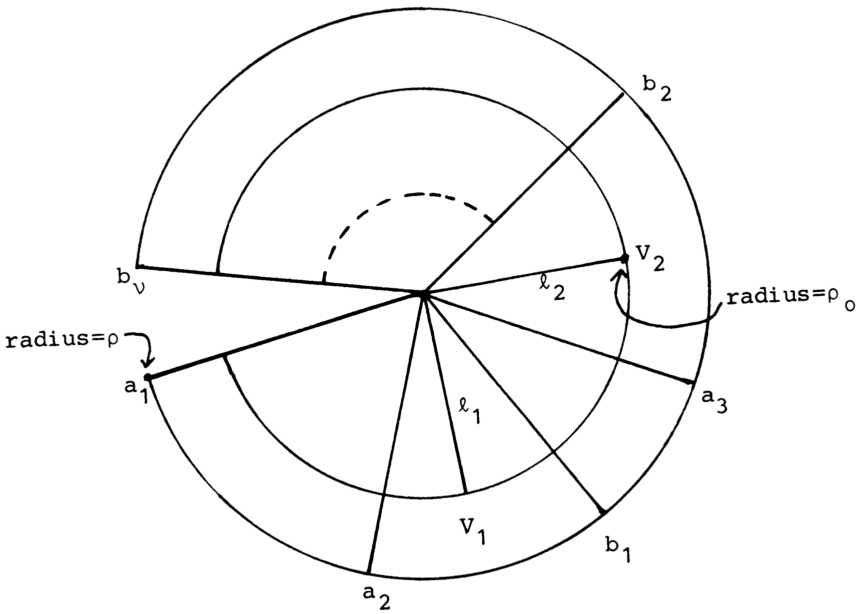

Figure 2

Let γ_j ($j = 1, 2, \ldots, \nu$) be the circular arcs which are defined by $\varepsilon = \rho_0 e^{i\xi}$ ($\alpha_{j-1} \leqslant \xi \leqslant \alpha_j$), respectively, where $\alpha_0 = a_1$, $\alpha_\nu = b_\nu$. Set, l_0: $\varepsilon = te^{ia_1}$ $(0 < t < \rho_0)$, l_ν: $\varepsilon = te^{ib_\nu}$ $(0 < t < \rho_0)$. Then $\gamma_1 + \gamma_2 + \cdots + \gamma_\nu = C = \{\varepsilon \colon |\varepsilon| = \rho_0, a_1 \leqslant \arg \varepsilon \leqslant b_\nu\}$. The boundaries of $\hat{S}_1, \hat{S}_2, \ldots, \hat{S}_\nu$ are, respectively, $l_{j-1} + \gamma_j - l_j, j = 1, 2, \ldots, \nu$.

For $\varepsilon \in \hat{S}_1 \cup \hat{S}_2 \cup \cdots \cup \hat{S}_\nu$, set $\delta(\varepsilon) = \delta_j(\varepsilon)$ if $\varepsilon \in \hat{S}_j$. Since

$$\frac{1}{2\pi i}\int_{l_{j-1}+\gamma_j-l_j} \frac{\delta_j(\xi)}{\xi - \varepsilon}\,d\xi = \begin{cases} \delta_j(\varepsilon), & \varepsilon \in \hat{S}_j, \\ 0, & \varepsilon \notin \hat{S}_j, \end{cases}$$

we have

$$\delta(\varepsilon) = \frac{1}{2\pi i}\sum_{j=1}^{\nu}\int_{l_{j-1}+\gamma_j-l_j} \frac{\delta_j(\xi)}{\xi - \varepsilon}\,d\xi \quad \text{in } \hat{S}_1 \cup \hat{S}_2 \cup \cdots \cup \hat{S}_\nu.$$

Utilizing $1/(\xi - \varepsilon) = \sum_{m=0}^{N} \xi^{-(m+1)}\varepsilon^m + \varepsilon^{N+1}/\xi^{N+1}(\xi - \varepsilon)$, we derive

$$\delta(\varepsilon) = \frac{1}{2\pi i}\sum_{m=0}^{N}\left\{\sum_{j=1}^{\nu}\int_{l_{j-1}+\gamma_j-l_j} \xi^{-(m+1)}\delta_j(\xi)\,d\xi\right\}\varepsilon^m + \left\{\frac{1}{2\pi i}\sum_{j=1}^{\nu}\int_{l_{j-1}+\gamma_j-l_j} \frac{\delta_j(\xi)}{\xi^{N+1}(\xi - \varepsilon)}\,d\xi\right\}\varepsilon^{N+1}. \tag{3.2}$$

Since $\delta(\varepsilon)$ is asymptotically zero as ε tends to zero in $\hat{S}_1 \cup \hat{S}_2 \cup \cdots \cup \hat{S}_\nu$, the first term of (3.2) must be zero. Hence

$$\delta(\varepsilon) = \left\{\frac{1}{2\pi i}\sum_{j=1}^{\nu}\int_{l_{j-1}+\gamma_j-l_j} \frac{\delta_j(\xi)}{\xi^{N+1}(\xi - \varepsilon)}\,d\xi\right\}\varepsilon^{N+1}.$$

Therefore, we arrive at the following formula:

$$\delta(\varepsilon) = \frac{1}{2\pi i}\left\{\int_{l_0} \frac{\delta_1(\xi)}{\xi^N(\xi - \varepsilon)}\,d\xi - \int_{l_\nu} \frac{\delta_\nu(\xi)}{\xi^N(\xi - \varepsilon)}\,d\xi + \sum_{j=1}^{\nu-1}\int_{l_j} \frac{\sigma_j(\xi)}{\xi^N(\xi - \varepsilon)}\,d\xi + \int_C \frac{\delta(\xi)}{\xi^N(\xi - \varepsilon)}\,d\xi\right\}\varepsilon^N$$

for $\varepsilon \in \hat{S}_1 \cup \hat{S}_2 \cup \cdots \cup \hat{S}_\nu$ and $N = 1, 2, \ldots$ where $\sigma_j = \delta_{j+1} - \delta_j$. Construct ν open sectors $\tilde{S}_1, \tilde{S}_2, \ldots, \tilde{S}_\nu$ such that the boundaries of $\tilde{S}_j$ are $\Omega_j \cup T_j \cup \Omega^j$, where

$$\Omega_j = \left\{\varepsilon \colon \varepsilon = |\varepsilon| e^{i(\alpha_{j-1}+\theta)}, |\varepsilon| \leqslant \rho_1\right\},$$

$$\Omega^j = \left\{\varepsilon \colon \varepsilon = |\varepsilon| e^{i(\alpha_j-\theta)}, |\varepsilon| \leqslant \rho_1\right\},$$

$$T_j = \left\{\varepsilon \colon \varepsilon = \rho_1 e^{i\beta}, \alpha_{j-1} + \theta \leqslant \beta \leqslant \alpha_j - \theta\right\}, \qquad j = 1, 2, \ldots, \nu,$$

$0 < \rho_1 < \rho_0$ and θ is a small positive number such that $\lambda\theta < \pi$ (cf. Figure 3).

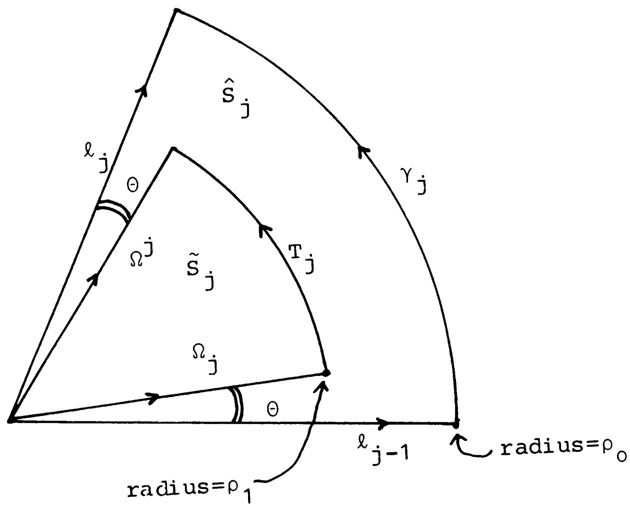

FIGURE 3

Then, $\tilde{S}_j \subset \hat{S}_j, j = 1, 2, \ldots, \nu$.

For $\varepsilon \in \tilde{S}_1 \cup \tilde{S}_2 \cup \cdots \cup \tilde{S}_\nu$,

$$\left|\int_C \frac{\delta(\xi)}{\xi^N(\xi-\varepsilon)}\,d\xi\right| \leqslant \int_C \frac{|\delta(\xi)|}{|\xi|^N|\xi-\varepsilon|}\,|d\xi| \leqslant \int_C \frac{K_\delta}{\rho_0^N(\rho_0-\rho_1)}\,|d\xi|$$
$$= \frac{K_\delta \rho_0}{\rho_0^N(\rho_0-\rho_1)}(b_\nu - a_1) = \frac{B_\delta}{\rho_0^{N-1}},$$

where $K_\delta = \max_{\xi\in C}|\delta(\xi)|$, $B_\delta = K_\delta(b_\nu - a_1)/(\rho_0-\rho_1)$.

$$\left|\int_{l_0} \frac{\delta_1(\xi)}{\xi^N(\xi-\varepsilon)}\,d\xi\right| = \left|\int_0^{\rho_0} \frac{\delta_1(te^{ia_1})}{t^N e^{iNa_1}(te^{ia_1}-\varepsilon)} e^{ia_1}\,dt\right| \leqslant \int_0^{\rho_0} \frac{|\delta_1(te^{ia_1})|}{t^{N+1}\sin\theta}\,dt$$
$$\leqslant \frac{M_0}{\sin\theta}\int_0^{\rho_0} t^{-N-1}\exp(-c_1/t^\lambda)\,dt$$
$$= \frac{M_0}{\lambda\sin\theta}\int_{1/\rho_0^\lambda}^{+\infty} \tau^{(N/\lambda)-1}\exp(-c_1\tau)\,d\tau$$
$$< \frac{M_0}{\lambda\sin\theta}\int_0^{+\infty} \tau^{(N/\lambda)-1}\exp(-c_1\tau)\,d\tau$$
$$= \frac{M_0}{\lambda\sin\theta} C_1^{-(N/\lambda)}\Gamma(N/\lambda).$$

Similarly,

$$\left|\int_{l_\nu} \frac{\delta_\nu(\xi)}{\xi^N(\xi-\varepsilon)}\,d\xi\right| < \frac{M_0}{\lambda\sin\theta} C_1^{-(N/\lambda)}\Gamma(N/\lambda),$$

and

$$\left|\int_{l_j} \frac{\sigma_j(\xi)}{\xi^N(\xi-\varepsilon)}\,d\xi\right| < \frac{M_0}{\lambda \sin\theta} C_1^{-(N/\lambda)}\Gamma(N/\lambda), \qquad j = 1,2,\ldots,\nu-1.$$

Since $\Gamma(N/\lambda) \leqslant M_1(N/\lambda)^{(N/\lambda)}e^{-(N/\lambda)}$ for some $M_1 > 0$, we have

$$\left|\int_{l_0} \frac{\delta_1(\xi)}{\xi^N(\xi-\varepsilon)}\,d\xi\right| < \frac{M_0 M_1}{\lambda \sin\theta}\left(\frac{N}{c_1\lambda}\right)^{(N/\lambda)} e^{-(N/\lambda)},$$

$$\left|\int_{l_\nu} \frac{\delta_\nu(\xi)}{\xi^N(\xi-\varepsilon)}\,d\xi\right| < \frac{M_0 M_1}{\lambda \sin\theta}\left(\frac{N}{c_1\lambda}\right)^{(N/\lambda)} e^{-(N/\lambda)}$$

and

$$\left|\int_{l_j} \frac{\sigma_j(\xi)}{\xi^N(\xi-\varepsilon)}\,d\xi\right| < \frac{M_0 M_1}{\lambda \sin\theta}\left(\frac{N}{c_1\lambda}\right)^{(N/\lambda)} e^{-(N/\lambda)}$$

for $\varepsilon \in \tilde{S}_1 \cup \cdots \cup \tilde{S}_\nu, j = 1,2,\ldots,\nu-1$. Then,

$$|\delta(\varepsilon)| \leqslant \frac{1}{2\pi}\left\{\frac{(\nu+1)M_0M_1}{\lambda\sin\theta}\left(\frac{N}{c_1\lambda}\right)^{(N/\lambda)} e^{-(N/\lambda)} + \frac{B_\delta}{\rho_0^{N-1}}\right\}|\varepsilon|^N$$

$$= \frac{M_2}{2\pi}\left\{1 + \frac{B_\delta\rho_0}{M_2}\left(\frac{c_1\lambda e}{N\rho_0^\lambda}\right)^{(N/\lambda)}\right\}\left(\frac{N|\varepsilon|^\lambda}{c_1\lambda}\right)^{(N/\lambda)} e^{-(N/\lambda)},$$

where $M_2 = (\nu+1)M_0M_1/\lambda\sin\theta > 0$.

Since $\{1 + (B_\delta\rho_0/M_2)(c_1\lambda e/N\rho_0^\lambda)^{(N/\lambda)}\} \to 1$ as $c_1 \to 0$, we have

$$|\delta(\varepsilon)| \leqslant M_3(\delta)\left(\frac{N|\varepsilon|^\lambda}{c_1\lambda}\right)^{(N/\lambda)} e^{-(N/\lambda)} \tag{3.3}$$

for $\varepsilon \in \tilde{S}_1 \cup \tilde{S}_2 \cup \cdots \cup \tilde{S}_\nu$, where $M_3(\delta)$ is a constant which is independent of c_1 as c_1 tends to zero.

For a given ε, choose N so that $N/\lambda < c_1/|\varepsilon|^\lambda \leqslant (N+1)/\lambda$. We have

$$\frac{N|\varepsilon|^\lambda}{\lambda c_1} < 1 \quad \text{and} \quad -\frac{c_1}{|\varepsilon|^\lambda} \geqslant -\frac{N+1}{\lambda}. \tag{3.4}$$

Then, it follows from (3.3) and (3.4) that we have

$$|\delta(\varepsilon)| \leqslant M_3(\delta)e^{1/\lambda}\exp\left(-c_1/|\varepsilon|^\lambda\right) \leqslant M_4\exp\left(-c_1/|\varepsilon|^\lambda\right),$$

where $M_4 = \max\{M_0, M_3(\delta)e^{1/\lambda}\}$ which is independent of c_1 as c_1 tends to zero. Choosing $l_1, l_2, \ldots, l_{\nu-1}$ in various ways, we can prove that

$$|\delta_j(\varepsilon)| \leqslant M_4\exp\left(-c_1/|\varepsilon|^\lambda\right) \quad \text{in } S_j', j = 2,\ldots,\nu-1,$$

where

$$S_j' = \{\varepsilon: a_j < \arg\varepsilon < b_j, 0 < |\varepsilon| < \rho_1\}, \qquad j = 1, 2, \ldots, \nu.$$

Since l_0 and l_ν are boundaries of S, they cannot be moved. To obtain a similar estimate in S_1', set $c_2 = c_1/\cos(\lambda\theta/2)$, then

$$|\delta_1(\varepsilon e^{i(a_1+\theta/2)})| \leq M_4 \exp(-c_2\cos(\lambda\theta/2)/|\varepsilon|^\lambda)$$

on

$$\{\varepsilon: \varepsilon = |\varepsilon|e^{i\theta/2}, 0 < |\varepsilon| < \rho_1\} \cup \{\varepsilon: \varepsilon = |\varepsilon|e^{-i\theta/2}, 0 < |\varepsilon| < \rho_1\}.$$

Let $h(\varepsilon) = \exp\{c_2/\varepsilon^\lambda\}$ ($\varepsilon \in \{\varepsilon: 0 < |\varepsilon| \leq \rho_1\}$). For $\arg\varepsilon = \theta/2$, $\arg\varepsilon = -\theta/2$, we have

$$|\delta_1(\varepsilon e^{i(a_1+\theta/2)})h(\varepsilon)| \leq M_4 \exp\{-c_2\cos(\lambda\theta/2)/|\varepsilon|^\lambda\} \cdot \exp\{c_2\cos(\lambda\theta/2)/|\varepsilon|^\lambda\} = M_4$$

and for $\varepsilon = |\varepsilon|e^{i\eta}$, $-\theta/2 < \eta < \theta/2$, we have

$$|\delta_1(\varepsilon e^{i(a_1+\theta/2)})h(\varepsilon)| \leq E\exp\{c_2\cos(\lambda\eta)/|\varepsilon|^\lambda\} \leq E\exp\{c_2/|\varepsilon|^\lambda\},$$

where E is a bound for $\delta_1(\varepsilon e^{i(a_1+\theta/2)})$ on $\{\varepsilon: -\theta/2 \leq \arg\varepsilon \leq \theta/2, 0 < |\varepsilon| \leq \rho_1\}$. Then, by the Phragmén-Lindelöf theorem, we have

$$|\delta_1(\varepsilon e^{i(a_1+\theta/2)})h(\varepsilon)| \leq M_4 \quad \text{for } \varepsilon \in \{\varepsilon: -\theta/2 < \arg\varepsilon < \theta/2, 0 < |\varepsilon| < \rho_1\}$$

i.e. $|\delta_1(\varepsilon e^{i(a_1+\theta/2)})| \leq M_4\exp\{-c_2\cos(\lambda\theta/2)/|\varepsilon|^\lambda\}$ for

$$\varepsilon = |\varepsilon|e^{i\eta} \in \{\varepsilon: -\theta/2 < \arg\varepsilon < \theta/2, 0 < |\varepsilon| < \rho_1\}.$$

Thus, $|\delta_1(\varepsilon)| \leq M_4\exp(-c_1/|\varepsilon|^\lambda)$ in S_1'; similarly, $|\delta_\nu(\varepsilon)| \leq M_4\exp(-c_1/|\varepsilon|^\lambda)$ in S_ν'. Choosing M as the maximal value of M_4 and a bound of $|\delta_j(\varepsilon)|\exp(c_1/\rho_1^\lambda)$ on $\{\varepsilon: a_j \leq \arg\varepsilon \leq b_j, \rho_1 \leq |\varepsilon| \leq \rho\}, j = 1, 2, \ldots, \nu$, which is independent of c_1 as c_1 tends to zero, we can obtain $|\delta_j(\varepsilon)| \leq M\exp(-c_1/|\nu\varepsilon|^\lambda)$ in S_j, $j = 1, 2, \ldots, \nu$.

(2) PROOF OF THEOREM 1. For each $\eta > 0$ we define an auxiliary function

$$h_\eta(\varepsilon) = \exp\{-\eta/\varepsilon^\lambda\} \qquad (\varepsilon \in \{\varepsilon: -\pi/2\alpha \leq \arg\varepsilon \leq \pi/2\alpha, 0 < |\varepsilon| \leq \rho\})$$

where $1 < \lambda < \alpha$. Set, $\delta_j(\varepsilon) = \phi_j(\varepsilon)h_\eta(\varepsilon)$ which depends on η, $j = 1, 2, \ldots, \nu$. Then, for $\varepsilon \in l_0$,

$$\begin{aligned} |\delta_1(\varepsilon)| &= |\phi_1(\varepsilon)h_\eta(\varepsilon)| \leq M_0\left|\exp\{-\eta/|\varepsilon|^\lambda e^{-\pi\lambda i/2\alpha}\}\right| \\ &= M_0\exp\{-\eta\cos(\lambda\pi/2\alpha)/|\varepsilon|^\lambda\}; \end{aligned}$$

for $\varepsilon \in l_\nu$,

$$\begin{aligned} |\delta_\nu(\varepsilon)| &= |\phi_\nu(\varepsilon)h_\eta(\varepsilon)| \leq M_0\left|\exp\{-\eta/|\varepsilon|^\lambda e^{\pi\lambda i/2\alpha}\}\right| \\ &= M_0\exp\{-\eta\cos(\lambda\pi/2\alpha)/|\varepsilon|^\lambda\}; \end{aligned}$$

and for $\varepsilon = |\varepsilon|e^{i\beta_j} \in S_j \cap S_{j+1}$,

$$|\delta_{j+1}(\varepsilon) - \delta_j(\varepsilon)| = |\phi_{j+1}(\varepsilon) - \phi_j(\varepsilon)|\,|h_\eta(\varepsilon)| \leq M_0\exp\{-\eta\cos(\lambda|\beta_j|)/|\varepsilon|^\lambda\}.$$

Since $0 < \lambda |\beta_j| < \lambda\pi/2\alpha$, we have $\cos(\lambda|\beta_j|) > \cos(\lambda\pi/2\alpha)$. Then we derive

$$|\delta_{j+1}(\varepsilon) - \delta_j(\varepsilon)| \leqslant M_0 \exp\left\{-\eta \cos(\lambda\pi/2\alpha)/|\varepsilon|^\lambda\right\} \quad \text{in } S_j \cap S_{j+1},$$

and $\delta_j(\varepsilon)$ is asymptotically zero as ε tends to zero in S_j. Set $c_1 = \eta\cos(\lambda\pi/2\alpha)$. Then by Theorem 2, there exists a positive number H which is independent of c_1 as c_1 tends to zero ($c_1 \to 0$, as $\eta \to 0$) such that

$$|\delta_j(\varepsilon)| = |\phi_j(\varepsilon)h_\eta(\varepsilon)| \leqslant M \exp\left\{-\eta\cos(\lambda\pi/2\alpha)/|\varepsilon|^\lambda\right\} \quad \text{in } S_j, j = 1, 2, \ldots, \nu.$$

As $\eta \to 0$, $h_\eta(\varepsilon) \to 1$ for every ε; so we conclude that $|\phi_j(\varepsilon)| \leqslant M$ in S_j, $j = 1, 2, \ldots, \nu$.

(3) Proof of Theorem 3. Let $h(\varepsilon) = \exp\{\mu/\varepsilon\}$, $\varepsilon \in S = \{\varepsilon\colon -\pi/2\alpha \leqslant \arg\varepsilon \leqslant \pi/2\alpha, 0 < |\varepsilon| \leqslant \rho\}$, and set $\phi_j(\varepsilon) = \delta_j(\varepsilon)h(\varepsilon)$, $j = 1, 2, \ldots, \nu$. Then, for $\varepsilon = |\varepsilon| e^{-\pi i/2\alpha}$,

$$|\phi_1(\varepsilon)| = |\delta_1(\varepsilon)|\,|h(\varepsilon)| \leqslant M_0 \exp\{-\mu \operatorname{Re}(1/\varepsilon)\} \exp\{\mu \operatorname{Re}(1/\varepsilon)\} = M_0;$$

similarly, for $\varepsilon = |\varepsilon| e^{\pi i/2\alpha}$, $|\phi_\nu(\varepsilon)| \leqslant M_0$; for $\varepsilon = |\varepsilon| e^{i\beta_j} \in S_j \cap S_{j+1}$, $|\phi_{j+1}(\varepsilon) - \phi_j(\varepsilon)| \leqslant M_0$; and for $\varepsilon = |\varepsilon| e^{i\xi_j} \in S_j$,

$$|\phi_j(\varepsilon)| = |\delta_j(\varepsilon)|\,|h(\varepsilon)| \leqslant A \exp\left\{\mu\cos\xi_j/|\varepsilon|\right\} \leqslant A\exp\{\mu/|\varepsilon|\}$$

where $A > 0$ is a bound for δ_j in S_j. By Theorem 1, we have $|\phi_j(\varepsilon)| \leqslant M$ in S_j ($j = 1, 2, \ldots, \nu$), i.e. $|\delta_j(\varepsilon)| \leqslant M \exp\{-\mu \operatorname{Re}(1/\varepsilon)\}$ in S_j, $j = 1, 2, \ldots, \nu$.

Acknowledgement. I would like to thank my advisor Professor Sibuya for his many valuable suggestions.

References

1. C.-H. Lin. *The sufficiency of Matkowsky-condition in the problem of resonance*, Trans. Amer. Math. Soc. (to appear).

2. W. Rudin, *Real and complex analysis*, McGraw-Hill, New York, 1974.

3. Y. Sibuya, *A theorem concerning uniform simplification at a transition point and the problem of resonance*, SIAM J. Math. Anal. **12** (1981), 653–668.

Department of Mathematics, College of Natural Science, Fujen University, Hsin Chuang 242, Taipei, Taiwan