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THREE CONVEX SETS

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ABSTRACT. We construct a Choquet simplex K such that there is a universally measurable affine function f on K, which satisfies the barycentric calculus, and is zero on the set of extreme points, but is not identically zero. We also construct a closed convex bounded set of a Banach space without extreme points, but such that each point is the barycenter of a maximal measure. Finally, we construct a closed bounded set L of $l^1(\mathbf{R})$ and a maximal measure on L which is supported by a weak Baire set which contains no extreme points.

1. Introduction. Let K be a convex compact set and $A(K)$ the space of affine continuous functions on K. For a probability measure μ on K, its barycenter b_μ is the unique point in K such that for each $f \in A(K)$, we have

$$f(b_\mu) = \int f \, d\mu. \tag{1}$$

A function f on K is said to satisfy the barycentric calculus if it is μ-measurable for each Radon measure μ on K and if (1) holds. Each point of K is the barycenter of a maximal measure μ on K (see [**1**, **4**] for references about all the claims concerning Choquet's theory) where maximal means maximal for the order

$$\mu \prec \nu \Leftrightarrow \forall f \text{ convex continuous on } K, \quad \mu(f) \leq \nu(f). \tag{2}$$

A maximal measure is in many respects "very close to the extreme points". For example, it is supported by each K_σ which contains the extreme points. It is hence a natural question whether a function which satisfies the barycentric calculus is determined by its values on the extreme points (this question is asked in [**3**]). In the next section we shall construct a counterexample where K is a Choquet simplex.

Now let E be a Banach space and A a closed convex bounded set of E. A Radon measure μ on A is said to be maximal if it is maximal for the order.

$$\mu \prec \nu \Leftrightarrow \forall f \text{ convex continuous on } A, \quad \mu(f) \leq \nu(f).$$

In §3, we shall construct a closed convex bounded set L in $l^1(\mathbf{R})$, a maximal measure μ on L, and a weak Baire set B with $\mu(B) = 1$, but such that B contains no extreme points. So, even though L has the Radon-Nikodym property, it is very different from a weak compact set. This construction settles a problem left open by G. A. Edgar [**2**]. Also related to the work of Edgar is an example of a closed

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bounded convex set of $c_0(\mathbf{R})$ with no extreme points, but such that each point is the barycenter of a maximal Radon measure. The construction of this example is somewhat similar to the construction of the first example.

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2. A big Choquet simplex. Let L be a compact space and A a subspace of $C(L)$ which contains the constants. The construction will follow by repeated use of a constant procedure which we describe now. We denote by λ the Lebesgue measure on $[0, 1]$. Let $L' = L \cup L \times [0, 1]$, provided with the "porcupine" topology, i.e. the topology given by the following subbasis of open sets:

$$\{l\} \times U \quad \text{for } l \in L,\ U \text{ open in } [0,1],$$
$$V \cup (V \setminus \{l\}) \times [0,1] \quad \text{for } l \in L,\ V \text{ open in } L.$$

Let j be the canonical injection $L \to L'$ and p: $L' \to L$ given by $p(l) = l$ and $p((l, t)) = l$.

Let A' be the space of $f' \in C(L')$ which satisfy the following conditions:

$$f' \circ j \in A, \tag{3}$$

$$\forall l \in L, \quad f'(l) = \int f'(l, t)\, d\lambda(t), \tag{4}$$

where for simplicity we write $f'(l, t)$ instead of $f'((l, t))$. Let i be the injection of A in A' given by $i(f) = f \circ p$.

LEMMA 1. *Let μ be a Radon probability on L'. If $p(\mu)$ is diffuse, then $\mu(j(L)) = 1$.*

PROOF. If $\mu(L' \setminus L) > 0$, since $L' \setminus L$ is the union of the open sets $\{l\} \times [0, 1]$, there exists $l \in L$ with $\mu(\{l\} \times [0, 1]) > 0$. Since $p(\{l\} \times [0, 1]) = \{l\}$, it shows that $p(\mu)(\{l\}) > 0$, and proves the lemma.

We also need a condition which will ensure that $A' = A(K')$ for a simplex K'. Recall that K is a simplex if and only if $A(K)$ satisfies the Riesz interpolation property.

DEFINITION 2. We say that A satisfies the property $(*)$ if, given $f_1, f_2, g_1, g_2 \in A$, with $f_1, f_2 < g_1, g_2$, and given n points $x_1, \ldots, x_n \in L$, $t_1, t_1', \ldots, t_2, t_2' \in \mathbf{R}$ with

$$\forall i \leq n, \quad \sup(f_1(x_i), f_2(x_i)) \leq t_i < t_i' \leq \inf(g_1(x_i), g_2(x_i)), \tag{5}$$

there exists $h \in A$, $f_1, f_2 < h < g_1, g_2$, and $\forall i \leq n$, $t_i < h(x_i) < t_i'$.

LEMMA 3. *If A satisfies $(*)$, then A' satisfies $(*)$.*

PROOF. Let $f_1', f_2', g_1', g_2' \in A'$ with $f_1', f_2' < g_1', g_2'$. Let ε be such that $\inf(g_1', g_2') - \sup(f_1', f_2') > \varepsilon$. Let $x_1, \ldots, x_n \in L'$. We can assume that $x_1, \ldots, x_m \in L$ and $x_{m+1}, \ldots, x_n \in L' \setminus L$ for one $m \leq n$. Let $t_i, t_i' \in \mathbf{R}$ such that

$$\forall i \leq n, \quad \sup(f_1'(x_i), f_2'(x_i)) \leq t_i < t_i' \leq \inf(g_1'(x_i), g_2'(x_i)).$$

It is easily seen from the definition of the topology of L' that there is a finite set $F \in L$ such that for $l \notin F$, and for g one of f_1', f_2', g_1', g_2', one has

$$\forall t \in [0,1], \quad |g(l) - g(l,t)| \leqslant \varepsilon/10. \tag{6}$$

For $l \in F$, let

$$t_l = \int \sup(f_1'(l,t), f_2'(l,t))\, d\lambda(y), \quad t_l' = \int \inf(g_1'(l,t), g_2'(l,t))\, d\lambda(t).$$

Since A satisfies (*), there is an $h \in A$ such that if f_1, f_2, g_1, g_2 denote the restrictions, respectively, of f_1', f_2', g_1', g_2' to L, we have $f_1 + \varepsilon/4, f_2 + \varepsilon/4 < h < g_1 - \varepsilon/4, g_2 - \varepsilon/4$ and $\forall i \leqslant n$, $t_i < h(x_i) < t_i'$; $\forall l \in L$, $t_l < h(l) < t_l'$. Let F' be the finite set of those $l \in L$ such that $\{l\} \times [0,1]$ contains one x_i $(m < i \leqslant n)$. For $l \notin F \cup F'$ we set $h'(l,t) = h(l)$. For $l \in F \cup F'$ it is easily seen that there is a $u \in C([0,1])$ such that

$$\forall t \in [0,1], \quad f_1'(l,t), f_2'(l,t) < u(t) < g_1'(l,t), g_2'(l,t),$$

$$\int u(t)\, d\lambda(t) = h(l),$$

$$\forall\, m < i \leqslant n \text{ such that } x_i = (l, a_i), \quad t_i < u(a_i) < t_i',$$

and we set $h'(l,t) = u(t)$. It is straightforward to check that $h' \in A'$ and satisfies all the requirements. The lemma is proved.

We now proceed to the construction. Let Ω be the first uncountable ordinal. For $\alpha \leqslant \Omega$, we shall construct compact spaces L_α and subspaces A_α of $C(L_\alpha)$ which satisfy (*). For $\beta < \alpha$, there is a map $p_{\beta,\alpha}$ from L_α into L_β, and an injection $j_{\alpha,\beta}$ of L_β into L_α, with $p_{\beta,\alpha} \circ j_{\alpha,\beta}$ being the identity of L_β.

The construction goes as follows. We start with $L_0 = \{0\}$, $A_0 = C(L_0) = \mathbf{R}$. Suppose the construction has been done for each $\alpha < \gamma$.

1*st case*. γ is of the form $\alpha + 1$. Then we set $L_\gamma = L_\alpha'$, $A_\gamma = A_\alpha'$. If j: $L \to L'$ and p: $L' \to L$ are the natural injection and projection, for $\beta < \alpha$ we set $j_{\gamma,\beta} = j \circ j_{\alpha,\beta}$ and $p_{\beta,\gamma} = p_{\gamma,\alpha} \circ p$. From Lemma 3, A_γ satisfies (*) and the construction is completed in this case.

2*nd case*. γ is a limit ordinal. Then L_γ is the projective limit of the $(L_\alpha)_{\alpha<\gamma}$. Mappings $j_{\gamma,\beta}$ and $p_{\beta,\gamma}$ are defined in the obvious way. For $f \in A_\beta$, let $i_\beta(f) = f \circ p_{\beta,\gamma}$ and let $A_\gamma = \bigcup_{\beta<\gamma} i_\gamma(A_\beta)$. All the verifications are straightforward, so the construction is completed.

Let $L = L_\Omega$, $A = \overline{A}_\Omega$. Since A_Ω satisfies (*), A also satisfies (*). Let $K = \{h \in A^*, h \geqslant 0, h(u) = 1\}$, where $u(l) = 1\ \forall l \in L$. Then, as is well known, $A = A(K)$. Moreover, L identifies as a subset of K, which contains the extreme points. Since A satisfies (*), A satisfies the Riesz interpolation property, so K is a Choquet simplex. For simplicity, we identify each L_α to a subset of L, and we denote by p_α the natural projection of L onto L_α.

Let K_α be the closed convex hull of L_α.

LEMMA 4. (a) *The set of extreme points of K is $E = L \setminus \bigcup_{\beta<\Omega} L_\beta$.*

(b) *For a diffuse Radon probability μ on L, there exists α with $\mu(L_\alpha) = 1$.*

(c) *Each Radon probability μ on L can be written in a unique way, $\mu = \mu' + \mu''$ where there is an α for which $\mu'(L_\alpha) = \mu'(L)$, and μ'' is atomic, $\mu''(E) = \mu''(L)$. In particular, E is universally measurable.*

(d) *Let μ_1 and μ_2 be probabilities on L. If $\mu_1(f) = \mu_2(f)$ for each $f \in A(K)$, then μ_1 and μ_2 have the same restriction to E.*

PROOF. (a) If $a \in L_\beta$, then condition (4) shows that a is the barycenter of a copy of Lebesgue measure on $\{a\} \times [0,1]$, so a is not extremal. It follows that each extreme point belongs to E. The converse follows from (d).

(b) Since μ is diffuse, for each n there is an $\alpha_n < \Omega$ such that $p_{\alpha_n}(\mu)$ has no atoms of weight $\geqslant n^{-1}$. If $\alpha = \sup \alpha_n$, then $p_\alpha(\mu)$ is diffuse. From Lemma 2 and by induction one proves that for $\gamma > \alpha$ we have $p_\gamma(\mu)(j_{\mu,\alpha}(L_\alpha)) = 1$, so $\mu(L_\alpha) = 1$.

(c) Let μ be a Radon probability on L, and $\alpha < \Omega$ such that $\mu(K_\alpha) = \sup_{\beta<\Omega} \mu(K_\beta)$. Let μ' be the restriction of μ to L_α, and $\mu'' = \mu - \mu'$. We have $\mu'(L_\beta) = 0$ for each β. It follows from (b) that μ' is atomic and then, of course, is supported by E.

(d) It is enough to show that if $a \in E$ then $\mu_2(\{a\}) \geqslant \mu_1(\{a\})$. Let

$$H = \{\alpha < \Omega;\, p_{\alpha+1}(a) \in L_{\alpha+1} \setminus L_\alpha\}.$$

If for $\alpha \geqslant \beta$ we have $\alpha \notin H$, we show by induction that $p_\alpha(a) \in j_{\alpha,\beta}(L_\beta)$, so $a \in L_\beta$, a contradiction. It follows that H is unbounded.

Let $\alpha \in H$. Let $p_{\alpha+1}(a) = (l', t')$, where $l' \in L_\alpha$. Consider a sequence (u_n) of $C([0,1])$ with $\int u_n(t)\,dt = 0$ and $u_n(t) \to 1$ if $t = t'$, and $u_n(t) \to 0$ otherwise. Let $f_n \in A_{\alpha+1}$ be given by $f_n(l) = 0$ for $l \in L_\alpha$, $f_n(l,t) = 0$ if $l \neq l'$, $f_n(l',t) = u_n(t)$. Since

$$p_{\alpha+1}(\mu_1)(f_n) = p_{\alpha+1}(\mu_2)(f_n)$$

for each n, this forces $p_{\alpha+1}(\mu_2)(l,t) \geqslant p_{\alpha+1}(\mu_1)(l,t)$, i.e. $\mu_2(p_{\alpha+1}^{-1}(p_{\alpha+1}(a))) \geqslant \mu_1(\{a\})$. But letting α grow in H to Ω, one gets $\mu_2(\{a\}) \geqslant \mu_1(\{a\})$. The lemma is proved.

For $a \in K$, property (d) shows that we can define $f(a) = \mu(L \setminus E)$, where μ is any measure on L of barycenter a.

If $a \in E$, we have $f(a) = 0$, since δ_a represents a. If $a \in K_\alpha$, a is the barycenter of a measure on L_α, so $f(a) = 1$.

It remains to show that f satisfies the barycentric calculus. Let ν be a Radon probability on K. The map φ from $M_+^1(L)$ onto K, which sends a measure on its barycenter, is continuous; so there is a Radon probability θ on $M_+^1(L)$ such that $\nu = \varphi(\theta)$. Let $\bar{\theta}$ be the probability on L given by

$$(**) \qquad \bar{\theta}(g) = \int_{M_+^1(L)} \mu(g)\, d\theta(\mu)$$

for $g \in C(L)$. If $g \in A(K)$, we have

$$g(b_{\bar\theta}) = \bar\theta(g) = \int \mu(g)\, d\theta(\mu) = \int g(b_\mu)\, d\theta(\mu)$$
$$= \int g(x)\, d\nu(x) = g(b_\nu),$$

so $b_{\bar\theta} = b_\nu$.

There exist α and a countable set $D \subset E$ such that $\bar\theta(K_\alpha \cup D) = 1$. Since (**) holds also for upper semicontinuous functions, we have $\mu(K_\alpha \cup D) = 1$ θ-almost everywhere, whence it follows that $\mu(K_\alpha) = \mu(L \setminus E)$ θ-a.e. so we have

$$f(b_\nu) = f(b_{\bar\theta}) = \theta(L \setminus E) = \theta(K_\alpha) = \int \mu(K_\alpha)\, d\theta(\mu)$$
$$= \int \mu(L \setminus E)\, d\theta(\mu) = \int f(b_\mu)\, d\theta(\mu) = \int f \circ \varphi(\mu)\, d\theta(\mu).$$

The proof shows that $f \circ \varphi$ is θ-measurable. It follows that f is ν-measurable and $\int f \circ \varphi(\mu)\, d\theta(\mu) = \int f(x)\, d\nu(x)$, which finishes the proof.

THEOREM 5. *There exists a Choquet simplex K and an affine f on K which satisfies the barycentric calculus, and is not identically zero, but vanishes on the extreme points.*

3. A big convex set. The construction here will also use the same standard process at each step.

We denote by K the unit disk of $\mathbf{R}^2$, and by λ the uniform measure on the unit circle. Then λ is maximal in K, and its barycenter is zero. A point $k \in K$ will be denoted by $(k(1), k(2))$.

Let A be a bounded closed convex subspace of $c_0(\Gamma)$ with $0 \in A$. Let $\Gamma' = \Gamma \cup (\{1,2\} \times A)$. We consider $c_0(\Gamma)$ as a subspace of $c_0(\Gamma')$. For $a \in A$, $k \in K$, we define $a * k$ by

$$\begin{aligned} a * k(\gamma') &= a(\gamma) && \text{if } \gamma' = \gamma \in \Gamma, \\ a * k(\gamma') &= k(n) && \text{if } \gamma' \text{ is of the form } (n, a), n \in \{1,2\}, \\ &= 0 && \text{otherwise.} \end{aligned}$$

Notice that $a = a * 0$.

We denote by A' the closed convex hull of the points $a * k$ for $a \in A$, $k \in K$. We denote by p the canonical projection of $c_0(\Gamma')$ onto $c_0(\Gamma)$, and by p_a the map from $c_0(\Gamma')$ to $\mathbf{R}^2$ given by $p_a(x) = (x((1, a)), x((2, a)))$. Notice that $A = p(A')$.

LEMMA 6. $A' = \{\sum_{a \in A} \alpha_a\, a * k_a;\ \alpha_a \geqslant 0, \sum \alpha_a \leqslant 1, k_a \in K\}$.

PROOF. Let p be the positive part of the unit ball of $l'(A)$. The map φ: $P \times K^A \to c_0(\Gamma')$ which sends $(\alpha_a)_{a \in A} \times (k_a)_{a \in A}$ to $\sum_{a \in A} \alpha_a a * k_a$ is continuous when $l^1(A)$ is provided with the w^*-topology, K^A with the product topology, $l^1(A) \times K^A$ with the product topology and $c_0(\Gamma')$ with the weak topology. Hence its image is weak compact. Since it is convex it is norm closed. The claim follows.

LEMMA 7. *Let μ be a maximal diffuse measure on A. Then μ, seen as a measure on A', is maximal.*

PROOF. Let ν be a measure on A' with $\nu \prec \nu$. Since $\mu = p(\mu) \prec p(\nu)$, we have $p(\nu) = \mu$.

Suppose, for contradiction, that $\nu(A' \setminus A) > 0$. Since ν is a Radon measure, the intersection of a family of closed sets of full measure is of full measure. Since $A = A' \cap c_0(\Gamma)$, there exists a and ε such that $\nu(U) > 0$, where $U = \{x \in A'; \|p_a(x)\| > \varepsilon\}$. Let ν' be the normalisation of the restriction of ν to U. For $x \in U$, $x = \sum \alpha_a a * k_a$, we have $p_a(x) = \alpha_a k_a$, so $\|p_a(x)\| > \varepsilon$, so $\alpha_a > \varepsilon$, so $\sum_{b \neq a} \alpha_b < 1 - \varepsilon$. It follows that $p(x) \in B = \varepsilon a + (1 - \varepsilon)A$; so $p(\nu')(B) = 1$.

Since $\nu' \ll \nu$, [**2**, Theorem 5-1] shows that ν' is maximal. However, $B \setminus \{a\}$ is a movable set in the sense of [**2**], since for $b \in B$ we have

$$b = 1/2((\gamma b + (1 - \gamma)a) + (2 - \gamma)b - (1 - \gamma)a),$$

where $\gamma = (1 - 2\varepsilon)/(1 - \varepsilon)$; this contradiction shows that $\nu(A) = 1$, so $p(\nu) = \nu = \mu$.

LEMMA 8. *Each point of a is the barycenter of a maximal diffuse measure μ on A.*

PROOF. Let λ_a be the image measure of λ under the map $k \to a * k$. It is a diffuse measure, with barycenter a. Moreover, it is straightforward from Lemma 6 to check that each point, $a * k$ with $\|k\| = 1$, is extremal; so λ_a is maximal.

We now have the tools for a first example:

THEOREM 9. *There exist a closed bounded convex set L of $l^1(\mathbf{R})$, a maximal measure μ on L, and a weak Baire set B of $l^1(\mathbf{R})$ which supports μ but contains no extreme points.*

PROOF. We make the above construction with $A = K$, $\Gamma = \{1,2\}$ and set $L = A'$. Then Lemma 6 shows that L is a subset of $l^1(\Gamma')$, and since the l^1-norm is finer than the c_0-norm, it is closed in l^1.

From Lemma 7, the measure λ on A is maximal on A'. Let $I = \{(n, a) \in \Gamma', a \in A, n = 1,2\}$ and $B = \{x \in l^1(\Gamma'); x(i) = 0, \forall\, i \in I\}$. Then $A \subset B$, so $\lambda(B) = 1$. Moreover, B contains no extreme point since each point of a is the barycenter of the measure λ_a.

Finally, since $l^1(I)$ is a subspace of l^∞, its dual is weak*-separable, so there is a sequence $y_n \in l^1(\Gamma')^*$ such that $B = \{x; \forall\, n, y_n(x) = 0\}$, so B is a weak Baire set. The proof is complete.

We can now proceed to the construction of the last example. For $\alpha < \Omega$ we construct sets Γ_α, with $\operatorname{card} \Gamma_\alpha = \operatorname{card} \mathbf{R}$, and closed convex sets $A_\alpha \subset c_0(\Gamma_\alpha)$ such that for $\beta < \alpha$, we have $\Gamma_\beta \subset \Gamma_\alpha$, and $A_\beta \subset A_\alpha$ (where $c_0(\Gamma_\beta)$ is identified to a subspace of $c_0(\Gamma_\alpha)$) in the following way. We set $A_0 = K$, $\Gamma_0 = \mathbf{N}$. If the construction has been done for α, we set $\Gamma_{\alpha+1} = \Gamma'_\alpha$ and $A_{\alpha+1} = A'_\alpha$. If the construction has been done for each $\alpha < \gamma$, and γ is limit, then $\Gamma_\gamma = \bigcup_{\alpha<\gamma} \Gamma_\alpha$ and A_γ is the closure of $\bigcup_{\alpha<\gamma} A_\alpha$.

We set $A = \bigcup_{\alpha<\Omega} A_\alpha$. Since each sequence in A is contained in one of the A_α, A is closed. Let $a \in A$. Then there is an α such that $a \in A_\alpha$. From Lemma 7, a is the barycenter of a maximal probability μ on $A'_{\alpha+1}$. Let ν be a probability on μ with $\mu \prec \nu$. Using Lemma 6, one shows by induction over γ that if p_γ denotes the natural projection of $c_0(\Gamma)$ onto $c_0(\Gamma_\gamma)$, then $p_\gamma(\nu) = \mu$. It follows that $\nu = \mu$, so μ is maximal on A. This also shows that A has no extreme points.

THEOREM 10. *There is a convex bounded closed set of $c_0(\mathbf{R})$ which has no extreme points, but such that each point is the barycenter of a maximal measure.*

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