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ON THE STRONG UNICITY OF BEST CHEBYSHEV APPROXIMATION OF DIFFERENTIABLE FUNCTIONS

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ABSTRACT. Let X be a normed linear space, U_n an n-dimensional Chebyshev subspace of X. For $f \in X$ denote by $p(f) \in U_n$ its best approximation in U_n. The problem of strong unicity consists in estimating how fast the nearly best approximants $g \in U_n$ satisfying $\|f - g\| \leq \|f - p(f)\| + \delta$ approach $p(f)$ as $\delta \to 0$. In the present note we study this problem in the case when $X = C^r[a, b]$ is the space of r-times continuously differentiable functions endowed with the supremum norm $(1 \leq r \leq \infty)$.

Introduction. Let X be a normed linear space and let U_n be an n-dimensional subspace of X. We say that $p \in U_n$ is a best approximation of $f \in X$ if $\|f - p\| = \operatorname{dist}(f, U_n) = \inf\{\|f - g\|: g \in U_n\}$. (Since U_n is finite dimensional each $f \in X$ possesses a best approximant.) Assume, in addition, that U_n is a Chebyshev subspace of X, i.e. each $f \in X$ has a unique best approximation in U_n. Let us denote by $p(f) \in U_n$ the best approximant of f. A question of considerable interest is that of how fast the "nearly best approximations" $g \in U_n$ satisfying

$$\|f - g\| \leq \|f - p(f)\| + \delta \tag{1}$$

approach $p(f)$ as $\delta \to 0$. This is the so-called strong unicity problem. In the case when, for a given $f \in X$ and any $g \in U_n$ satisfying (1) with some $0 < \delta \leq 1$, the relation $\|g - p(f)\| \leq C(f)\delta^\gamma$ holds, we say the degree of strong unicity at f is γ $(0 < \gamma \leq 1$, $C(f) > 0$ is independent of $g)$.

Consider the classical case of Chebyshev approximation, when $X = C[a, b]$ is the space of real or complex continuous functions endowed with the supremum norm $\|f\|_C = \max\{|f(x)|: x \in [a, b]\}$. The famous Haar-Kolmogorov theorem states that $U_n \subset C[a, b]$ is a Chebyshev subspace of $C[a, b]$ if and only if each nontrivial element of U_n has at most $n - 1$ distinct zeros at $[a, b]$. The finite-dimensional Chebyshev subspaces of $C[a, b]$ are usually called Haar spaces.

The strong unicity problem for $C[a, b]$ was solved by D. Newman and H. Shapiro **[5]**. They proved that if U_n is a Haar space then for each $f \in C[a, b]$ the degree of strong unicity at f is $\frac{1}{2}$ in the complex case and 1 in the real case. (For the L_p spaces the problem of strong unicity was solved in **[1]** for $1 < p < \infty$ and in **[3]** for $p = 1$.)

Now let $X = C^r[a, b] = \{f \in C[a, b]: f^{(r)} \in C[a, b]\}$ be the space of r-times continuously differentiable functions on $[a, b]$ endowed with the same supremum norm $(1 \leq r \leq \infty)$. The characterization of Chebyshev subspaces of $C^r[a, b]$ was

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given in [**2** and **4**]. (In [**2**] the Chebyshev subspaces of $C^r[a, b]$ were studied in the real case. Later in [**4**] using another approach we characterized the Chebyshev subspaces of $C^r[a, b]$ in the complex case.) It turned out that the characteristic property of Chebyshev subspaces of $C^r[a, b]$ (which is actually independent of $1 \leqslant r \leqslant \infty$) is more general than the Haar property.

The aim of this paper is to give a solution to the problem of strong unicity in $C^r[a, b]$.

Preliminaries. We start with a lemma which characterizes the best approximants of $f \in C[a, b]$.

LEMMA [**6**, P. 178]. *Let U_n be an n-dimensional subspace of $C[a, b]$, $f \in C[a, b]$. Then $p \in U_n$ is a best approximation of f if and only if there exist points $a \leqslant x_1 < x_2 < \cdots < x_m \leqslant b$, where $1 \leqslant m \leqslant n + 1$ in the real case and $1 \leqslant m \leqslant 2n + 1$ in the complex case, and numbers $a_i \neq 0$, $1 \leqslant i \leqslant m$, such that*

$$f(x_i) - p(x_i) = (\bar{a}_i/|a_i|)\|f - p\|_C \qquad (1 \leqslant i \leqslant m) \tag{2}$$

and

$$\sum_{i=1}^{m} a_i g(x_i) = 0 \tag{3}$$

for any $g \in U_n$.

DEFINITION 1. The set of points $a \leqslant x_1 < \cdots < x_m \leqslant b$, where $1 \leqslant m \leqslant n + 1$ in the real case and $1 \leqslant m \leqslant 2n + 1$ in the complex case, is called an extremal set of the n-dimensional subspace $U_n \subset C[a, b]$ if there exist numbers $a_i \neq 0$, $1 \leqslant i \leqslant m$ (real or complex, respectively) such that (3) holds for every $g \in U_n$. The numbers $\{a_i\}_{i=1}^m$ are called coefficients of the extremal set $\{x_i\}_{i=1}^m$. (Note that the coefficients of an extremal set are, in general, defined nonuniquely.)

It can be shown that a necessary and sufficient condition for U_n to be a Haar space is that no nontrivial element of U_n can vanish on an extremal set of U_n. This statement can be considered as another definition of Haar spaces.

DEFINITION 2. Let U_n be an n-dimensional subspace of $C^1[a, b]$. Then U_n is called a semi-Haar space if and only if there does not exist an extremal set $\{x_i\}_{i=1}^m$ of U_n with coefficients $\{a_i\}_{i=1}^m$ and $g \in U_n \setminus \{0\}$ such that $g(x_i) = 0$ for any $1 \leqslant i \leqslant m$ and $\operatorname{Re} a_i g'(x_i) = 0$ if $x_i \in (a, b)$ $(1 \leqslant i \leqslant m)$. (*Remark*: In the real case the relation $\operatorname{Re} a_i g'(x_i) = 0$ is equivalent to $g'(x_i) = 0$.)

It turned out (see [**4**]) that $U_n \subset C^r[a, b]$ is a Chebyshev subspace of $C^r[a, b]$ if and only if U_n is a semi-Haar space $(1 \leqslant r \leqslant \infty)$. Therefore it is natural to consider the problem of strong unicity of best Chebyshev approximation of differentiable functions with respect to semi-Haar spaces. (It is shown in [**4**] that the family of semi-Haar spaces is essentially wider than that of Haar spaces, in particular there are different spaces of real and complex lacunary polynomials, i.e. polynomials with gaps, which do not satisfy the Haar property but are nevertheless semi-Haar spaces.)

New results. In this section we shall present some strong unicity type results for semi-Haar spaces. Let us give several additional notations. For $f \in C[a, b]$ we denote by

$$\omega(f, h) = \sup\{|f(x_1) - f(x_2)| : x_1, x_2 \in [a, b], |x_1 - x_2| \leqslant h\}$$

the modulus of continuity of f $(0 < h \leqslant b - a)$. Furthermore, set

$$\mathrm{Lip}\, \alpha = \{f \in C[a, b]: \sup h^{-\alpha}\omega(f, h) < \infty\}, \qquad 0 < \alpha \leqslant 1.$$

Let $\varphi_1, \ldots, \varphi_n$ be a basis in $U_n \subset C^1[a, b]$ and consider the modulus $\omega^*(h) = \sum_{i=1}^{n} \omega(\varphi_i', h)$. Then for any $p \in U_n$,

$$\omega(p', h) \leqslant K_0 \|p\|_C \omega^*(h), \tag{4}$$

where $K_0 > 0$ is independent of p. If U_n is a semi-Haar space then $p(f) \in U_n$ denotes the unique best approximation of $f \in C^1[a, b]$.

THEOREM 1. *Let U_n be a semi-Haar space. Then for any given $f \in C^1[a, b]$ and every $g \in U_n$ such that*

$$\|f - g\|_C \leqslant \|f - p(f)\|_C + \delta(\omega(f', \delta) + \omega^*(\delta) + \delta) \tag{5}$$

with some $0 < \delta \leqslant b - a$, we have

$$\|p(f) - g\|_C \leqslant K_1(\omega(f', \delta) + \omega^*(\delta) + \delta), \tag{6}$$

where the constant $K_1 > 0$ is independent of g and δ.

In particular, if $f', \varphi_1', \ldots, \varphi_n' \in \mathrm{Lip}\, \alpha$, where $\varphi_1, \ldots, \varphi_n$ are the basis functions of U_n, then the degree of strong unicity at f is $\alpha/(1 + \alpha)$. Thus for semi-Haar spaces the degree of strong unicity depends, in general, on the constructive properties of the functions considered. On the other hand, Theorem 1 implies that in $C^2[a, b]$ the degree of strong unicity (with respect to semi-Haar spaces) is $\frac{1}{2}$ for each $f \in C^2[a, b]$.

COROLLARY. *Let $U_n \subset C^2[a, b]$ be a semi-Haar space. Then for each $f \in C^2[a, b]$ the degree of strong unicity at f is equal to $\frac{1}{2}$.*

The above corollary shows the surprising fact that in $C^2[a, b]$ the Newman-Shapiro theorem (complex version) remains true under the weaker semi-Haar assumption.

On the other hand, as the following theorem shows in $C^1[a, b]$, the degree of strong unicity for non-Haar spaces is, in general, less than $\frac{1}{2}$.

THEOREM 2. *Assume $U_n \subset C^1[a, b]$ does not satisfy the Haar property, i.e. some $g \in U_n \backslash \{0\}$ has n distinct zeros at $[a, b]$. Then for any $0 < \alpha \leqslant 1$ there exists a function $f_\alpha \in C^1[a, b]$ such that $f_\alpha' \in \mathrm{Lip}\, \alpha$, 0 is a best approximant of f_α in U_n, and*

$$\|f_\alpha - \delta^{\alpha/(\alpha+1)} g\|_C \leqslant \|f_\alpha\|_C + K_2\delta \qquad (0 < \delta \leqslant 1) \tag{7}$$

where $K_2 \geqslant 0$ is independent of δ. Moreover, if $\alpha = 1$ then $f_1 \in C^\infty[a, b]$.

Thus the degree of strong unicity given by Theorem 1 is sharp, in general, and cannot be improved even for "very smooth" functions or real functions.

PROOF OF THEOREM 1. Let $f \in C^1[a, b] \setminus U_n$ (the case $f \in U_n$ is trivial) and assume that $g \in U_n$, $g \neq p(f)$ satisfies (5) with some $0 < \delta \leqslant b - a$. Set $f_0 = f - p(f)$, $g_0 = g - p(f)$ ($g_0 \in U_n \setminus \{0\}$). Then 0 is the best approximation of f_0 and, by (5),

$$\|f_0 - g_0\|_C \leqslant \|f_0\|_C + \delta A(\delta), \tag{8}$$

where $A(h) = \omega(f', h) + \omega^*(h) + h$ ($0 < h \leqslant b - a$). Furthermore, we have, by (4),

$$\begin{aligned}\omega(f_0', h) &\leqslant \omega(f', h) + \omega(p'(f), h) \leqslant \omega(f', h) + K_0\|p(f)\|_C\omega^*(h) \\ &\leqslant c_1 A(h).\end{aligned} \tag{9}$$

(Here and in what follows the positive constants depending only on f, a, b and U_n are denoted by c_i, $i = 1, 2, \ldots$.)

Since 0 is the best approximant of f_0 in U_n, it follows by the Lemma that there exists an extremal set $\{x_i\}_{i=1}^m$ of U_n with coefficients $\{a_i\}_{i=1}^m$ such that

$$f_0(x_i) = (\bar{a}_i/|a_i|)\|f_0\|_C \qquad (1 \leqslant i \leqslant m). \tag{10}$$

Hence, by (8),

$$\begin{aligned}|f_0(x_i) - g_0(x_i)|^2 &= |\|f_0\|_C - (a_i/|a_i|)g_0(x_i)|^2 \\ &= \|f_0\|_C^2 - (2\|f_0\|_C/|a_i|)\operatorname{Re} a_i g_0(x_i) + |g_0(x_i)|^2 \\ &\leqslant \|f_0 - g_0\|_C^2 \leqslant (\|f_0\|_C + \delta A(\delta))^2 \leqslant \|f_0\|_C^2 + c_2\delta A(\delta) \qquad (1 \leqslant i \leqslant m).\end{aligned} \tag{11}$$

This obviously implies that

$$\operatorname{Re} a_i g_0(x_i) \geqslant -(c_2|a_i|/2\|f_0\|_C)\delta A(\delta) \geqslant -c_3\delta A(\delta) \qquad (1 \leqslant i \leqslant m). \tag{12}$$

Moreover, by definition of extremal sets, $\sum_{i=1}^m a_i g_0(x_i) = 0$. Hence, by (12), for any $1 \leqslant j \leqslant m$,

$$\operatorname{Re} a_j g_0(x_j) = -\sum_{\substack{i=1 \\ i \neq j}}^{m} \operatorname{Re} a_i g_0(x_i) \leqslant c_4\delta A(\delta).$$

Thus we finally obtain that

$$|\operatorname{Re} a_i g_0(x_i)| \leqslant c_5\delta A(\delta) \qquad (1 \leqslant i \leqslant m). \tag{13}$$

Furthermore, using this inequality in (11) we have

$$|g_0(x_i)| \leqslant \sqrt{c_6\delta A(\delta)} \leqslant \sqrt{c_6}\, A(\delta) = c_7 A(\delta) \qquad (1 \leqslant i \leqslant m). \tag{14}$$

Consider an arbitrary point x_i of the extremal set belonging to (a, b). Set

$$f_1(x) = (1/|a_i|)\operatorname{Re} a_i f_0(x), \qquad g_1(x) = (1/|a_i|)\operatorname{Re} a_i g_0(x).$$

Then the real functions $f_1, g_1 \in C^1[a, b]$ have the following properties:

$$f_1(x_i) = \|f_0\|_C = \|f_1\|_C, \qquad f_1'(x_i) = 0; \tag{15}$$

$$\|f_1 - g_1\|_C \leqslant \|f_0 - g_0\|_C \leqslant \|f_0\|_C + \delta A(\delta) = \|f_1\|_C + \delta A(\delta); \tag{16}$$

$$|g_1(x_i)| \leqslant c_8\delta A(\delta), \tag{17}$$

where (15), (16) and (17) follow immediately from (10), (8) and (13), respectively.

Using equations (16), (15) and (9) we obtain that for any real h such that $|h| \leq \min\{x_i - a, b - x_i\}$,

$$(18) \quad \begin{aligned} g_1(x_i + h) &\geq f_1(x_i + h) - \|f_1 - g_1\|_C \geq f_1(x_i + h) - \|f_1\|_C - \delta A(\delta) \\ &= -\delta A(\delta) + f_1(x_i + h) - f_1(x_i) \geq -\delta A(\delta) - |h|\omega(f_1', |h|) \\ &\geq -\delta A(\delta) - |h|\omega(f_0', |h|) \geq -\delta A(\delta) - c_1|h|A(|h|). \end{aligned}$$

Furthermore for some $\xi \in [a, b]$ such that $|\xi - x_i| \leq |h|$, we have

$$g_1'(x_i)h = g_1(x_i + h) - g_1(x_i) + h(g_1'(x_i) - g_1'(\xi)).$$

This, together with (18), (17), (4) and (8), yields

$$\begin{aligned} g_1'(x_i)h &\geq g_1(x_i + h) - g_1(x_i) - |h|\omega(g_1', |h|) \\ &\geq -\delta A(\delta) - c_1|h|A(|h|) - c_8\delta A(\delta) - |h|\omega(g_0', |h|) \\ &\geq -c_9\delta A(\delta) - c_1|h|A(|h|) - K_0\|g_0\|_C\omega^*(|h|)|h| \\ &\geq -c_9\delta A(\delta) - c_1|h|A(|h|) - K_0(2\|f_0\|_C + (b - a)A(b - a))|h|\omega^*(|h|) \\ &\geq -c_9\delta A(\delta) - c_{10}|h|A(|h|) \geq -c_{11}(\delta A(\delta) + |h|A(|h|)). \end{aligned}$$

Since the above inequality holds for any h such that $|h|$ is small enough, we may set $h = \pm c_{12}\delta$ (with appropriate $0 < c_{12} < 1$) which gives us two estimations:

$$g_1'(x_i) \geq -c_{11}(\delta A(\delta) + c_{12}\delta A(c_{12}\delta))/c_{12}\delta \geq -c_{13}A(\delta),$$
$$g_1'(x_i) \leq c_{11}(\delta A(\delta) + c_{12}\delta A(c_{12}\delta))/c_{12}\delta \leq c_{13}A(\delta).$$

Hence $|g_1'(x_i)| \leq c_{13}A(\delta)$, i.e.

$$(19) \qquad |\operatorname{Re} a_i g_0'(x_i)| \leq c_{14}A(\delta)$$

for those $1 \leq i \leq m$ for which x_i belong to (a, b).

Let us consider, on U_n, the functional

$$F(p) = \max_{1 \leq i \leq m} |p(x_i)| + \max_{\substack{1 \leq i \leq m \\ x_i \in (a, b)}} |\operatorname{Re} a_i p'(x_i)| \qquad (p \in U_n).$$

Since $\{x_i\}_{i=1}^m$ is an extremal set of U_n with coefficients $\{a_i\}_{i=1}^m$, and U_n is a semi-Haar space, it follows by definition of semi-Haar spaces that $F(p) > 0$ for every $p \in U_n \backslash \{0\}$. Moreover, it can be easily seen that F is continuous on U_n (endowed with the supremum norm). Hence by compactness of the unit ball in U_n,

$$(20) \qquad \|p\|_C \leq c_{15}F(p)$$

for each $p \in U_n$. Furthermore, by (14) and (19),

$$F(g_0) \leq (c_7 + c_{14})A(\delta).$$

Combining the above estimation with (20) we finally obtain

$$\|g - p(f)\|_C = \|g_0\|_C \leq c_{15}F(g_0) \leq c_{15}(c_7 + c_{14})A(\delta).$$

The proof of Theorem 1 is completed.

REMARK. Actually the Chebyshev property of semi-Haar spaces was not used in the proof of Theorem 1. Therefore Theorem 1 implies, in particular, that semi-Haar spaces are Chebyshev subspaces of $C^1[a, b]$.

PROOF OF THEOREM 2. Let $x_1, \ldots, x_n \in [a, b]$ be distinct zeros of $g \in U_n \setminus \{0\}$. Consider the linear functionals $L_i \in U_n^*$ given by $L_i(p) = p(x_i)$ $(p \in U_n)$, $1 \leq i \leq n$. Evidently, L_i, $1 \leq i \leq n$, are linearly dependent on U_n (otherwise they would span U_n^*, which is impossible since $g(x_i) = 0$, $1 \leq i \leq n$). This immediately implies $\{x_i\}_{i=1}^n$, or a proper subset of it, is an extremal set of U_n. Hence, without loss of generality, we may assume $\{x_i\}_{i=1}^m$ is an extremal set of U_n $(1 \leq m \leq n)$ with corresponding nonzero coefficients $\{a_i\}_{i=1}^m$, and $g(x_i) = 0$, $1 \leq i \leq m$.

There exists a function $\tilde{F} \in C^\infty[a, b]$ such that $\tilde{F}(x_i) = \bar{a}_i/|a_i|$ $(1 \leq i \leq m)$ and $\|\tilde{F}\|_C = 1$.

Set

$$f_\alpha(x) = \tilde{F}(x)\left(1 - \gamma \prod_{i=1}^{m} |x - x_i|^{1+\alpha}\right), \qquad 0 < \alpha \leq 1, \tag{21}$$

where $\gamma > 0$ is chosen so that $\gamma \prod_{i=1}^m |x - x_i|^{1+\alpha} \leq 1$, i.e. $\|f_\alpha\|_C = 1$. Since $f_\alpha(x_i) = \tilde{F}(x_i) = \bar{a}_i/|a_i|$ $(1 \leq i \leq m)$ it follows by the Lemma that 0 is a best approximant of f_α. Moreover, it can be easily seen that $f_\alpha' \in \operatorname{Lip} \alpha$ $(0 < \alpha < 1)$ and $f_1 \in C^\infty[a, b]$.

Let us estimate the quantity $\|f_\alpha - \delta^{\alpha/(\alpha+1)} g\|_C$. Since $g(x_i) = 0$, $1 \leq i \leq m$, we can obtain, for any $x \in [a, b]$,

$$|g(x)| \leq (\|\operatorname{Re} g'\|_C + \|\operatorname{Im} g'\|_C) \min_{1 \leq i \leq m} |x - x_i|. \tag{22}$$

Furthermore, it can be easily shown that

$$\prod_{i=1}^{m} |x - x_i|^{1+\alpha} \geq c_{16}\left(\min_{1 \leq i \leq m} |x - x_i|\right)^{1+\alpha} \qquad (x \in [a, b]),$$

where $c_{16} > 0$ does not depend on x. Hence, by (21) and (22), for any $x \in [a, b]$,

$$\begin{aligned}
|f_\alpha(x) - \delta^{\alpha/(\alpha+1)} g(x)| &\leq |f_\alpha(x)| + \delta^{\alpha/(\alpha+1)} |g(x)| \\
&\leq 1 - \gamma \prod_{i=1}^{m} |x - x_i|^{1+\alpha} + \delta^{\alpha/(\alpha+1)} (\|\operatorname{Re} g'\|_C + \|\operatorname{Im} g'\|_C) \min_{1 \leq i \leq m} |x - x_i| \\
&\leq 1 - c_{17}\left(\min_{1 \leq i \leq m} |x - x_i|\right)^{1+\alpha} + c_{18} \delta^{\alpha/(\alpha+1)} \min_{1 \leq i \leq m} |x - x_i| \\
&\leq \max_{\xi > 0} \left(1 - c_{17} \xi^{1+\alpha} + c_{18} \delta^{\alpha/(\alpha+1)} \xi\right) = 1 + c_{19} \delta.
\end{aligned}$$

Thus $\|f_\alpha - \delta^{\alpha/(\alpha+1)} g\|_C \leq 1 + c_{19}\delta$ for any $\delta > 0$.

The proof of Theorem 2 is completed.

Note that the strong unicity type result given by Theorem 1 can be applied as usual to the study of modulus of continuity of the operator of best approximation and discretization errors.

REFERENCES

1. B. O. Björnestål, *Local Lipschitz continuity of the metric projection operator*, Banach Center Publications, 4, Approximation Theory, PWN, Warsaw, 1979, pp. 43–53.

2. A. L. Garkavi, *Dimensionality of polyhedra of best approximation for differentiable functions*, Izv. Akad. Nauk SSSR Ser. Mat. **23** (1959), 93–114. (Russian)

3. A. Kroó, *On strong unicity of L_1-approximation*, Proc. Amer. Math. Soc. **83** (1981), 725–729.

4. ______, *On the unicity of best Chebyshev approximation of differentiable functions*, Acta Sci. Math. Szeged (to appear).

5. D. Newman and H. Shapiro, *Some theorems on Čebyšev approximation*, Duke Math. J. **30** (1963), 673–681.

6. I. Singer, *Best approximation in normed linear spaces by elements of linear subspaces*, Springer-Verlag, Berlin and New York, 1970.

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