

THE RIEMANN PROBLEM FOR NONCONVEX SCALAR CONSERVATION LAWS AND HAMILTON-JACOBI EQUATIONS

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ABSTRACT. We present a closed form expression for the viscosity solution to the Riemann problem for any scalar nonconvex conservation law. We then define an analogous problem for any scalar nonconvex Hamilton-Jacobi equation and obtain its (even simpler) solution. Extensions to two (and by inference, higher) space dimensional problems, when the initial discontinuity lies on a hyperplane, are also given.

We shall be concerned with solutions to the following nonlinear scalar partial differential equations:

$$(H-J) \quad \varphi_t + f(\varphi_x) + g(\varphi_y) = 0$$

and

$$(CL) \quad u_t + f(u)_x + g(u)_y = 0,$$

for $t > 0$, with special initial data.

This restriction to two space dimensions is made for simplicity only.

A rather complete existence and uniqueness theory for the Cauchy problem for (CL) has been known for some time (see, e.g. [2]). Recently a parallel theory was developed for (H-J) [1]. (More general equations have also been treated by both of these theories.)

Solutions of (CL) are well known to generally develop discontinuities, even if the initial data are smooth. Solutions of (H-J) stay continuous, but generally develop jumps in derivatives. To obtain uniqueness and the rest of the theory mentioned above, it suffices to consider only viscosity solutions.

A solution to (1) is a viscosity solution if it is the local uniform limit of $\varphi^\varepsilon(x, y, t)$ as $\varepsilon \downarrow 0$, where φ^ε solves

$$(1) \quad \varphi_t^\varepsilon + f(\varphi_x^\varepsilon) + g(\varphi_y^\varepsilon) = \varepsilon(\varphi_{xx}^\varepsilon + \varphi_{yy}^\varepsilon) + G_\varepsilon(x, y, t, \varphi^\varepsilon, \varphi_t^\varepsilon, \varphi_x^\varepsilon, \varphi_y^\varepsilon).$$

Here, G_ε converges uniformly to zero on compact subsets of $R^2 \times R^+ \times R^4$.

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A viscosity solution to (2) is the local L^1 limit of $u^\epsilon(x, y, t)$, as $\epsilon \downarrow 0$, where u^ϵ solves

$$(2) \quad u_t^\epsilon + f(u^\epsilon)_x + g(u^\epsilon)_y = \epsilon(u_{xx}^\epsilon + u_{yy}^\epsilon).$$

All solutions obtained below will be viscosity solutions.

The Riemann problem for (CL) in one space dimension is the following. Solve

$$(3) \quad u_t + f(u)_x = 0,$$

for $t > 0$, with initial data

$$u(x, 0) \equiv u^L, \quad x \leq 0 \quad \text{and} \quad u(x, 0) \equiv u^R, \quad x > 0,$$

for given constants u^L, u^R .

We have recently [4] obtained a simple formula for the solution to this problem for general f . For completeness, we shall present the formula here, and outline its derivation below.

The solution to (3) is well known to have the following properties (see e.g. [3]):

(i) $u(x, t) = u(x/t) = u(\zeta)$,

(ii) $u(\zeta)$ lies between u^L and u^R ,

(iii) $u(\zeta) \equiv u^L$, if $\zeta < -N$, $u(\zeta) \equiv u^R$, if $\zeta > N$, for some $N > 0$.

We have

LEMMA 1. *If $u^L < u^R$, then*

$$(4a) \quad \zeta u(\zeta) - f(u(\zeta)) = \max_{v \in [u^L, u^R]} [\zeta v - f(v)],$$

or if $u^L > u^R$, then

$$(4b) \quad \zeta u(\zeta) - f(u(\zeta)) = \min_{v \in [u^R, u^L]} [\zeta v - f(v)].$$

It is easy to see that the above Legendre transforms are uniformly Lipschitz continuous, with Lipschitz constant $\max[|u^L|, |u^R|]$, and are convex and concave, respectively, as functions of ζ . Taking the distribution derivative of equations (4) gives us an expression for $u(\zeta)$:

THEOREM 1. *If $u^L < u^R$, then*

$$(5a) \quad u(\zeta) = \frac{d}{d\zeta} \left(\max_{v \in [u^L, u^R]} [\zeta v - f(v)] \right).$$

If $u^L > u^R$, then

$$(5b) \quad u(\zeta) = \frac{d}{d\zeta} \left(\min_{v \in [u^R, u^L]} [\zeta v - f(v)] \right).$$

The analogous problem for (H-J) in one space dimension is

$$(6) \quad \varphi_t + f(\varphi_x) = 0,$$

for $t > 0$, with initial data

$$\varphi(x, 0) \equiv xu^L, \quad x \leq 0 \quad \text{and} \quad \varphi(x, 0) \equiv xu^R, \quad x > 0,$$

and given constants u^L, u^R .

We have

THEOREM 2.

$$\begin{aligned}
 (7) \quad \varphi(x, t) &= t[\zeta u(\zeta) - f(u(\zeta))] \\
 &= t \left(\max_{v \in [u^L, u^R]} [\zeta v - f(v)] \right), \quad \text{if } u^L < u^R, \\
 &= t \left(\min_{v \in [u^R, u^L]} [\zeta v - f(v)] \right), \quad \text{if } u^L > u^R.
 \end{aligned}$$

This last formula points out an interesting link between the Riemann problems for (CL) and (H-J).

In two space dimensions, we restrict ourselves to the following problems:

$$(8) \quad u_t + f(u)_x + g(u)_y = 0,$$

for $t > 0$, with initial data

$$u(x, y, 0) \equiv u^L, \quad \text{if } x \sin \theta - y \cos \theta \leq 0.$$

$$u(x, y, 0) \equiv u^R, \quad \text{if } x \sin \theta - y \cos \theta > 0.$$

$$(9) \quad \varphi_t + f(\varphi_x) + g(\varphi_y) = 0,$$

for $t > 0$, with initial data

$$\varphi(x, y, 0) \equiv (x \sin \theta - y \cos \theta)u^L + (x \cos \theta + y \sin \theta)u^M, \quad \text{if } x \sin \theta - y \cos \theta \leq 0.$$

$$\varphi(x, y, 0) \equiv (x \sin \theta - y \cos \theta)u^R + (x \cos \theta + y \sin \theta)u^M, \quad \text{if } x \sin \theta - y \cos \theta > 0.$$

Here u^L, u^M, u^R and θ are fixed constants, with $0 \leq \theta < \pi$.

For (CL), we gave piecewise constant initial data, with a single jump across a plane through the origin. For (H-J), we prescribed (φ_x, φ_y) to be constant, except for a single jump in normal derivative across a plane through the origin. We also require that φ be continuous, and, for convenience only, set $\varphi(0, 0, 0) = 0$.

We define

$$\zeta = \frac{x \sin \theta - y \cos \theta}{t}$$

and state

THEOREM 3. *The solution to (8) is*

$$(10a) \quad u(\zeta) = \frac{d}{d\zeta} \left(\max_{v \in [u^L, u^R]} [\zeta v - (f(v)\sin \theta - g(v)\cos \theta)] \right), \quad \text{if } u^L < u^R,$$

or

$$(10b) \quad u(\zeta) = \frac{d}{d\zeta} \left(\min_{v \in [u^R, u^L]} [\zeta v - (f(v)\sin \theta - g(v)\cos \theta)] \right), \quad \text{if } u^L > u^R.$$

THEOREM 4. *The solution to (9) is*

$$(11a) \quad \varphi(x, y, t) = u^M(x \cos \theta + y \sin \theta) + t \left(\max_{v \in [u^L, u^R]} (\zeta v - f(v \sin \theta + u^M \cos \theta) - g(-v \cos \theta + u^M \sin \theta)) \right), \quad \text{if } u^L < u^R,$$

or

$$(11b) \quad \varphi(x, y, t) = u^M(x \cos \theta + y \sin \theta) + t \left(\min_{v \in [u^R, u^L]} (\zeta v - f(v \sin \theta + u^M \cos \theta) - g(-v \cos \theta + u^M \sin \theta)) \right), \quad \text{if } u^L > u^R.$$

PROOFS. We begin by outlining the proof of Lemma 1, for $\zeta = 0$. It is known that the viscosity solution of (3) is a nondecreasing function of its initial data. In [4], we showed that this fact implies that the function $h(u^R, u^L) = f(u(0))$ is a nonincreasing function of its first argument, and nondecreasing function of its second.

For any u between u^L and u^R , we consider

$$\begin{aligned} \operatorname{sgn}(u^R - u^L)[h(u^R, u^L) - f(u)] &= -\operatorname{sgn}(u^L - u)[h(u^R, u^L) - h(u^R, u)] \\ &\quad + \operatorname{sgn}(u^R - u)[h(u^R, u) - h(u, u)] = \text{I} + \text{II}. \end{aligned}$$

By our previous remarks, both I and II are nonpositive, hence

$$\operatorname{sgn}(u^R - u^L)[f(u(0)) - f(u)] \leq 0,$$

for all u between u^R and u^L . Since $u(0)$ lies in this range, we have Lemma 1, for $\zeta = 0$.

To obtain Lemma 1 for all ζ , we make a change of independent variables

$$y = x - st, \quad \tau = t.$$

The resulting problem is still (3), with $f(u)$ replaced by $f(u) - su$. The previous argument gives us Lemma 1, for $\zeta = s$.

Theorem 1 follows from taking the distribution derivative of (4) (see [4]).

To verify Theorem (2), we first compute

$$\varphi_t = \zeta u(\zeta) - f(u(\zeta)) - tu(\zeta)\zeta_t = -f(u(\zeta)), \quad \varphi_x = tu(\zeta)\zeta_x = u(\zeta).$$

Thus, equation (6) is satisfied.

Letting $t \downarrow 0$ shows us that the initial data is taken on.

It only remains to show that φ is a viscosity solution. Since $\varphi_x = u(\zeta)$, we have, for $\bar{x} < -Nt$

$$\begin{aligned} \varphi(x, t) &= \bar{x}u^L - tf(u^L) + \int_{\bar{x}}^x u\left(\frac{s}{t}\right) ds \\ &= \bar{x}u^L - tf(u^L) + \lim_{\epsilon \downarrow 0} \int_{\bar{x}}^x u^\epsilon(s, t) ds \\ &= \lim_{\epsilon \downarrow 0} \left[\bar{x}u^L - tf(u^L) + \int_{\bar{x}}^x u^\epsilon(s, t) ds \right] \\ &= \lim_{\epsilon \downarrow 0} \varphi^\epsilon(s, t). \end{aligned}$$

Here

$$\varphi_t^\epsilon + f(\varphi_x^\epsilon) - \epsilon\varphi_{xx}^\epsilon = f(u^\epsilon(\bar{x}, t)) - f(u^L) - \epsilon u_x^\epsilon(\bar{x}, t).$$

It is easy to see that the function of t , defined on the right above, converges uniformly to zero on compact subsets of $0 < t < \bar{x}/N$, as $\epsilon \downarrow 0$. Thus φ is a viscosity solution. \square

To prove Theorem 3, we make an orthogonal change of variables

$$\begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} \sin \theta & -\cos \theta \\ \cos \theta & \sin \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sin \theta & \cos \theta \\ -\cos \theta & \sin \theta \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix}.$$

The solution u^ϵ , to the regularized problem solves

$$u_t^\epsilon + (\sin \theta f(u^\epsilon) - \cos \theta g(u^\epsilon))_z + (\cos \theta f(u^\epsilon) + \sin \theta g(u^\epsilon))_w = \epsilon(u_{zz}^\epsilon + u_{ww}^\epsilon)$$

for $t > 0$, with initial data

$$u^\epsilon(z, w, 0) \equiv u^L, \quad \text{if } z \leq 0 \quad \text{and} \quad u^\epsilon(z, w, 0) \equiv u^R, \quad \text{if } z > 0.$$

It is clear that the solution will be independent of w , and hence, by virtue of Theorem 1, will converge as $\epsilon \downarrow 0$, to (10).

For the proof of Theorem 4, we note that the same change of variables gives us

$$\varphi_t + f(\varphi_z \sin \theta + \varphi_w \cos \theta) + g(\varphi_z(-\cos \theta) + \varphi_w \sin \theta) = 0,$$

$$\varphi(z, w, 0) = u^L z + u^M w, \quad \text{if } z \leq 0 \quad \text{and} \quad \varphi(z, w, 0) = u^R z + u^M w, \quad \text{if } z > 0.$$

Let $\varphi = \psi + u^M w$. Then ψ satisfies

$$\psi_t + f(\psi_z \sin \theta + u^M \cos \theta) + g(\psi_z(-\cos \theta) + u^M \sin \theta) = 0,$$

$$\psi(z, w, 0) \equiv u^L z, \quad \text{if } z \leq 0 \quad \text{and} \quad \psi(z, w, 0) \equiv u^R z, \quad \text{if } z > 0.$$

Then, since ψ is obviously independent of w , we may use Theorem 2 to obtain the viscosity solution for this one-dimensional problem. This leads us to (11). It is easy to show that the resulting function, φ , is a viscosity solution for (9). We merely use a simple modification of the argument used in the proof of Theorem 2.

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