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A GAP TAUBERIAN THEOREM FOR GENERALISED ABSOLUTE ABEL SUMMABILITY

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ABSTRACT. A gap Tauberian theorem for generalised absolute Abel summability $|A_\alpha|$ is proved using Mel'nik's theorem on convolution transforms.

1. Introduction. The well-known gap Tauberian theorem for Abel summability $(A_0) \equiv (A)$ is a special case of the high indices theorem of Hardy and Littlewood [**2**, Theorem 114]. The gap Tauberian theorem for (A_α) summability has been proved by Krishnan [**3**]. Zygmund [**5**] has proved that $|A, \lambda_n|$ summability implies absolute convergence when (λ_n) satisfies the high indices condition $\lambda_{n+1}/\lambda_n \geqslant c > 1$, and Mel'nik [**4**] had deduced the same result as a corollary of his general theorem, which is stated here as Lemma 1. The gap Tauberian theorem for absolute Abel summability $|A_0| \equiv |A|$ is a special case of Zygmund's result when we take (λ_n) as a sequence of integers. The purpose of this note is to show that the gap Tauberian theorem for absolute A_α summability $|A_\alpha|$ can be deduced from Mel'nik's theorem.

2. Definitions and notations. Let $\alpha > -1$. For a given series $\sum_{n=0}^{\infty} a_n$, write

$$A_n = \sum_{r=0}^{n} a_r \qquad (n \geqslant 0),$$

$$\bar{A}(y) = \sum_{n \leqslant y} a_n,$$

$$a(x) = \sum_{n=0}^{\infty} a_n \binom{n+\alpha}{\alpha} x^n \qquad (0 < x < 1),$$

$$A(x) = \sum_{n=0}^{\infty} A_n \binom{n+\alpha}{\alpha} x^n \qquad (0 < x < 1),$$

$$f(x) = (1-x)^{\alpha+1} A(x) \qquad (0 < x < 1).$$

We assume that the series defining $a(x)$ and $A(x)$ converge for $0 < x < 1$. $\sum a_n$ is summable (A_α) to A if

$$f(x) = (1-x)^{\alpha+1} A(x) \to A \quad \text{as } x \to 1-,$$

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and absolutely A_α summable to A ($|A_\alpha|$ summable to A) if

$$\int_0^1 \left| \frac{d}{dx} f(x) \right| dx < \infty$$

and

$$f(x) = (1-x)^{\alpha+1} A(x) \to A \quad \text{as } x \to 1-.$$

Abel summability and absolute Abel summability are, respectively, A_0 and $|A_0|$ summability methods.

We say, following Mel'nik, that the function $s(v)$ belongs to the class $|T_0|$ if (i) $s(v)$ is of bounded variation in every finite interval; (ii) there exist constants μ, δ and a function $\theta(v)$, depending only on v, such that $0 < \mu, \delta \leqslant 1$, and the inequality

$$\operatorname{Re}\{e^{i\theta} ds(u)\} \geqslant \mu |ds(u)|$$

is satisfied for all u in $[v-\delta, v+\delta]$.

3. Theorem and lemmas. The main result is

THEOREM. *If Σa_n is summable $|A_\alpha|$ and satisfies a gap condition*

(G) *$a_n = 0$ for $n \neq n_k$, where (n_k) is a sequence of positive integers such that $n_0 > 0$, $n_{k+1}/n_k \geqslant c > 1$ for all $k = 0, 1, 2, \ldots$,*

then Σa_n converges absolutely.

We first prove some preliminary lemmas.

LEMMA 1. *Let the kernel $k(u)$ of the integral transform*

(1) $$g(v) = \int_{-\infty}^{\infty} k(v-u)\, ds(u)$$

be Borel measurable,

(2) $$\sum_{n=-\infty}^{\infty} \sup_{n \leqslant u < n+1} |k(u)| < \infty,$$

and its Fourier transform

(3) $$K(t) = \int_{-\infty}^{\infty} e^{-iut} k(u)\, du \neq 0$$

for $t \in E = (-\infty, \infty)$. Also assume $s(u)$ belongs to the class $|T_0|$ and

(4) $$\int_v^{v+1} |ds(u)| \leqslant M \text{ for } v \in E.$$

Then we can find a constant C depending only on δ, μ and $k(u)$ such that

(5) $$\int_{-\infty}^{\infty} |ds(v)| \leqslant C \int_{-\infty}^{\infty} |g(v)|\, dv.$$

This is contained in Mel'nik's theorem. [**4**, Theorem 1].

LEMMA 2. *If Σa_n is summable $|A_\alpha|$ and satisfies the gap condition* (G), *then*

(6) $$\int_0^1 |f'(x)|\, dx = \int_{-\infty}^{\infty} |g_1(v)|\, dv,$$

where

$$g_1(v) = \int_{-\infty}^{\infty} k_1(v, u)\, ds(u), \tag{7}$$

$$s(u) = \bar{A}(e^u), \tag{8}$$

$$k_1(v, u) = \begin{cases} \dfrac{\{1 - \exp(e^{-v})\}^{\alpha} e^{-v} \exp\{-e^{u-v}\}\Gamma(\alpha + e^u + 1)}{\Gamma(1+\alpha)\Gamma(e^u)} & \text{for } u \geqslant 0, \\ k(v-u) & \text{for } u < 0, \end{cases} \tag{9}$$

and

$$k(t) = \exp\{-(\alpha+1)t - e^{-t}\}/\Gamma(\alpha+1). \tag{10}$$

PROOF. First we find an expression for $f'(x)$ in terms of a_n.

$$\begin{aligned} f'(x) &= -(\alpha+1)(1-x)^{\alpha} \sum_{n=0}^{\infty} \binom{n+\alpha}{\alpha} A_n x^n + (1-x)^{\alpha+1} \sum_{n=0}^{\infty} \binom{n+\alpha}{\alpha} A_n n x^{n-1} \\ &= (1-x)^{\alpha}(\alpha+1)\left[-\sum_{n=0}^{\infty} \binom{n+\alpha}{\alpha} A_n x^n + (1-x)\sum_{n=1}^{\infty}\binom{n+\alpha}{n-1} A_n x^{n-1}\right] \\ &= (1-x)^{\alpha}(\alpha+1)\left[-\sum_{n=0}^{\infty} \binom{n+\alpha}{\alpha} A_n x^n \right.\\ &\qquad\qquad \left. + \sum_{n=1}^{\infty}\binom{n+\alpha}{n-1} A_n x^{n-1} - \sum_{n=1}^{\infty}\binom{n+\alpha}{n-1} A_n x^{n}\right] \\ &= (1-x)^{\alpha}(\alpha+1)\sum_{n=0}^{\infty}\left[-A_n\left\{\binom{n+\alpha}{\alpha} + \binom{n+\alpha}{n-1}\right\} + A_{n+1}\binom{n+1+\alpha}{n}\right] x^n \\ &= (1-x)^{\alpha}(\alpha+1)\sum_{n=0}^{\infty}\binom{n+1+\alpha}{n} a_{n+1}x^n \\ &= (1-x)^{\alpha}(\alpha+1)\sum_{n=1}^{\infty}\binom{n+\alpha}{n-1} a_n x^{n-1} \\ &= (1-x)^{\alpha}(\alpha+1)\int_1^{\infty} \frac{\Gamma(y+\alpha+1)}{\Gamma(y)\Gamma(\alpha+2)} x^{y-1}\, d\bar{A}(y) \\ &= (1-x)^{\alpha}\int_1^{\infty} \frac{\Gamma(y+\alpha+1)}{\Gamma(y)\Gamma(\alpha+1)} x^{y-1}\, d\bar{A}(y). \end{aligned}$$

Substituting $x = \exp\{-e^{-v}\}$, $y = e^u$, we obtain

$$f'(x) = F(v) = \int_0^{\infty} \frac{\{1 - \exp(-e^{-v})\}^{\alpha}\Gamma(\alpha + e^u + 1)}{\Gamma(1+\alpha)\Gamma(e^u)} \frac{\exp(-e^{u-v})}{\exp(-e^{-v})} d\bar{A}(e^u).$$

Hence,

$$\int_0^1 |f'(x)| dx = \int_{-\infty}^{\infty} |F(v)| \frac{dx}{dv} dv = \int_{-\infty}^{\infty} |g_1(v)|\, dv,$$

where

$$g_1(v) = F(v)\frac{dx}{dv}$$

$$= \int_0^\infty \frac{\{1-\exp(-e^{-v})\}^\alpha \Gamma(\alpha+e^u+1)}{\Gamma(1+\alpha)\Gamma(e^u)} \frac{\exp(-e^{u-v})}{\exp(-e^{-v})} \exp(-e^{-v})e^{-v}\, d\bar{A}(e^u)$$

$$= \int_0^\infty \frac{\{1-\exp(-e^{-v})\}^\alpha e^{-v}}{\Gamma(1+\alpha)\Gamma(e^u)} \exp\{-e^{u-v}\}\Gamma(\alpha+e^u+1)\, d\bar{A}(e^u)$$

$$+ \int_{-\infty}^0 k(v-u)\, d\bar{A}(e^u),$$

because $a_0 = 0$ under the assumption of the gap condition (G), and hence $\bar{A}(e^u) = 0$ for $u < 0$. Here $k(t)$ is given by (10). Hence,

$$\int_0^1 |f'(x)|dx = \int_{-\infty}^\infty k_1(v,u)\, ds(u),$$

where $s(u)$ and $k_1(v,u)$ are given by (8) and (9). □

Mel'nik [4, p. 834] has observed that his theorem can be applied to functions $g_1(v)$ which are expressible as

$$g_1(v) = \int_{-\infty}^\infty k_1(v,u)\, ds(u),$$

where $k_1(v,u)$ is not of the canonical form $k(v-u)$ as in (1) but can be "approximated" to a canonical form in a certain sense. The context of Lemma 3 below is that the kernel $k_1(v,u)$ appearing in (7) can be approximated in this sense. In the proof of the Theorem in §4 we incorporate the details as to how this approximation is useful.

LEMMA 3. *If $k(t)$ and $k_1(v,u)$ are the kernels of Lemma 2 given by* (10) *and* (9), *respectively, and* $-1 < \alpha < 0$, *then*

$$\sum_{n=-\infty}^{\infty} \max_{n \leqslant u < n+1} \int_{-\infty}^\infty |k_1(v,u) - k(v-u)|dv = L < \infty. \tag{11}$$

PROOF. For $u < 0$, $k_1(v,u) = k(v-u)$ and therefore it suffices to prove

$$\sum_{n=0}^{\infty} \max_{n \leqslant u < n+1} \int_{-\infty}^\infty |k_1(v,u) - k(v-u)|dv = L < \infty.$$

Let $u \geqslant 0$. Then

$$k_1(v,u) - k(v-u) = \frac{\{1-\exp(-e^{-v})\}^\alpha e^{-v}\exp\{-e^{u-v}\}\Gamma(\alpha+e^u+1)}{\Gamma(1+\alpha)\Gamma(e^u)}$$

$$- \frac{\exp\{(\alpha+1)(u-v) - e^{u-v}\}}{\Gamma(1+\alpha)}$$

$$= \frac{(1-x)^\alpha(\log x^{-1})x^y\Gamma(\alpha+y+1)}{\Gamma(1+\alpha)\Gamma(y)} - \frac{y^{\alpha+1}(\log x^{-1})^{\alpha+1}x^y}{\Gamma(1+\alpha)}$$

(where $y = e^u$, $x = \exp\{-e^{-v}\}$, $y \geqslant 1$)

$$= x^y\left(\log\frac{1}{x}\right)\left[\frac{(1-x)^\alpha\Gamma(\alpha+y+1)}{\Gamma(y)\Gamma(1+\alpha)} - \frac{(\log x^{-1})^\alpha y^{\alpha+1}}{\Gamma(1+\alpha)}\right]. \tag{12}$$

Now

$$\Gamma(\alpha+y+1)/\Gamma(y) = y^{\alpha+1} + O(y^{\alpha}) \text{ uniformly in } y \geqslant 1.$$

Also

$$\log x^{-1} = (1-x) + O(1-x)^2 \text{ uniformly in } \delta < x < 1$$

and, since $\alpha < 0$, it follows that

$$(\log x^{-1})^{\alpha} = (1-x)^{\alpha} + O(1-x)^{\alpha+1}$$

uniformly in $0 < x < 1$. It follows that, uniformly in $y \geqslant 1$, $0 < x < 1$, (12) is

$$x^{y} \log x^{-1}\left\{O\left(y^{\alpha}(1-x)^{\alpha}\right) + O\left(y^{\alpha+1}(1-x)^{\alpha+1}\right)\right\}.$$

Hence, uniformly in $y \geqslant 1$, we have

$$\begin{aligned}\int_{-\infty}^{\infty} |k_1(v,u) - k(v-u)|dv &= O\left\{y^{\alpha}\int_0^1 x^{y-1}(1-x)^{\alpha}\,dx\right\}\\ &\quad + O\left\{y^{\alpha+1}\int_0^1 x^{y-1}(1-x)^{\alpha+1}\,dx\right\}\\ &= O(1/y) = O(e^{-u}).\end{aligned}$$

The lemma follows.

4. Proof of the Theorem. By Theorems 2 and 5 of [**1**], summability $|A_{\lambda}|$ implies $|A_{\mu}|$ for $\lambda > \mu > -1$. Hence it suffices to prove the theorem for $-1 < \alpha < 0$. Define

$$g(v) = \int_{-\infty}^{\infty} k(v-u)\,ds(u) \tag{13}$$

and apply Lemma 1 to $g(v)$. It has the canonical form (1). (2) can be easily verified. The Fourier transform of $k(x)$ is

$$K(t) = \Gamma(\alpha+1+it)/\Gamma(\alpha+1) \neq 0$$

for any t, hence (3) is satisfied. If (G) is satisfied, and if $\delta > 0$ is sufficiently small, it follows that for any v the interval $[v-\delta, v+\delta]$ contains not more than one point at which the function $s(u) = \bar{A}(e^{u})$ has a jump. Hence we see that $s(u)$ belongs to the class $|T_0|$ where $\mu = 1$, δ depends only on c. Since $|A_{\alpha}|$ summability implies (A_{α}) summability and the gap Tauberian theorem is true for (A_{α}) summability (vide [**3**]), Σa_n is convergent and hence follows the boundedness of the terms of Σa_n. In view of (G), (4) now follows. (2) and (4) show that the integral (13) is absolutely convergent. All the requirements of Lemma 1 are satisfied and hence by the conclusion of the same lemma, we obtain

$$\sum_{n=0}^{\infty} |a_n| \leqslant C\int_{-\infty}^{\infty} |g(v)|\,dv. \tag{14}$$

Now

$$\int_{-\infty}^{\infty} |g(v)|\,dv \leqslant \int_{-\infty}^{\infty} |g(v) - g_1(v)|\,dv + \int_{-\infty}^{\infty} |g_1(v)|\,dv,$$

where $g_1(v)$ is given by (7). Then, substituting for $g(v)$ and $g_1(v)$ and applying Lemma 2, we get

$$\int_{-\infty}^{\infty} |g(v)|\,dv \leqslant \int_{-\infty}^{\infty} dv \int_{-\infty}^{\infty} |k_1(v, u) - k(v-u)|\,|ds(u)| + \int_0^1 |f'(x)|\,dx$$

$$\leqslant \int_{-\infty}^{\infty} |ds(u)| \int_{-\infty}^{\infty} |k_1(v, u) - k(v-u)|\,dv + \int_0^1 |f'(x)|\,dx$$

$$\leqslant \sum_{n=-\infty}^{\infty} \int_n^{n+1} |ds(u)| \max_{n \leqslant u < n+1} \int_{-\infty}^{\infty} |k_1(v, u) - k(v-u)|\,dv + \int_0^1 |f'(x)|\,dx$$

$$< M \sum_{n=-\infty}^{\infty} \max_{n \leqslant u < n+1} \int_{-\infty}^{\infty} |k_1(v, u) - k(v-u)|\,dv + \int_0^1 |f'(x)|\,dx$$

(since (4) is satisfied)

$$= ML + \int_0^1 |f'(x)|\,dx \quad \text{(by Lemma 3)}$$

$$(15) \quad = C' + \int_0^1 |f'(x)|\,dx.$$

From (14) and (15), we obtain

$$\sum_{n=0}^{\infty} |a_n| < C\left[C' + \int_0^1 |f'(x)|\,dx\right] = C_1 + C\int_0^1 |f'(x)|\,dx < \infty.$$

The theorem is proved.

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REFERENCES

1. D. Borewein and J. H. Rizvi, *On Abel type methods of summability*, J. Reine Angew. Math. **247** (1971), 139–147.
2. G. H. Hardy, *Divergent series*, Oxford, England, 1949.
3. V. K. Krishnan, *Gap Tauberian theorem for generalised Abel summability*, Math. Proc. Cambridge Philos. Soc. **78** (1975), 497–500.
4. V. I. Mel'nik, *Tauberian theorem in the metric of L for convolution transforms*, Ukrain. Mat. Ž. **32** (1980), 831–836.
5. A. Zygmund, *On certain integrals*, Trans. Amer. Math. Soc. **55** (1944), 170–204.

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