QUATERNION KAehlerian Manifolds
Isometrically Immersed in Euclidean Space

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Abstract. Let \( M \) be a complete \( 4n \)-dimensional quaternion Kaehlerian manifold isometrically immersed in the \((4n + d)\)-dimensional Euclidean space. In this note we prove that if \( d < n \), then \( M \) is a Riemannian product \( Q^m \times P \), where \( Q^m \) is the \( m \)-dimensional quaternion Euclidean space \((m \geq n - d)\) and \( P \) is a Ricci flat quaternion Kaehlerian manifold.

1. Introduction. In [2] C. Fwu obtains some topological restrictions for Kaehler manifolds which can be isometrically immersed in Euclidean space with low codimension. He essentially adapts a technique used by J. D. Moore [5] and uses elementary Morse theory. In a second result, he assumes the Kaehler manifold has nonnegative sectional curvature and thus obtains more information about its geometric structure (see [2, Theorem 2]).

In this paper we consider quaternion Kaehler manifolds isometrically immersed in Euclidean space with low codimension. We adapt the technique used in [5]. Since any quaternion Kaehler manifold with quaternion dimension \( n > 1 \) is an Einstein space (see [3, Theorem 3.3]), we obtain our main result without using any hypothesis about sectional curvature (compare [2, Theorems 1 and 2]).

The Main Theorem. Let \( M \) be a complete quaternion Kaehler manifold of real dimension \( 4n \). If \( \phi: M \to E_4^{4n+d} \) \((d < n)\) is an isometric immersion, then \( M = Q^m \times P \), the Riemannian product of \( Q^m \) \((m \geq n - d)\) and a Ricci flat quaternion Kaehler manifold \( P \) of real dimension \( 4(n - m) \). Moreover \( \phi = 1 \times \psi \), the product of the identity map of \( Q^m \) onto \( E_4^{4m} \) and an isometric immersion of \( P \) into \( E_4^{4(n-m)+d} \) (where \( E_r \) (resp. \( Q_r \)) denotes the \( r \)-dimensional Euclidean space (resp. the \( r \)-dimensional quaternion Euclidean space)).

2. The fundamental lemma. Let \( M \) be a quaternion Kaehler manifold, i.e. there exists a 3-dimensional vector bundle \( V \) of tensors of type \( (1, 1) \) with local basis of almost Hermitian structures \( I, J, K \) such that: (i) \( I \cdot J = -J \cdot I = K \), and (ii) for any local cross-section \( \Omega \) of \( V \) and any local vector field \( X \) tangent to \( M \), \( \nabla_X \Omega \) is also a local cross-section of \( V \), where \( \nabla \) denotes the covariant differentiation on \( M \) (see [3] for details).
Let \( \langle ., . \rangle \) be the Riemannian metric on \( M \) and \( R \) its curvature tensor. Then we have, \cite{3}

\[
\begin{align*}
\langle R(X, Y)IZ, IW \rangle - \langle R(X, Y)Z, W \rangle &= -C(X, Y)\langle KZ, W \rangle - B(X, Y)\langle JZ, W \rangle, \\
\langle R(X, Y)JZ, JW \rangle - \langle R(X, Y)Z, W \rangle &= -C(X, Y)\langle KZ, W \rangle - A(X, Y)\langle IZ, W \rangle, \\
\langle R(X, Y)KZ, KW \rangle - \langle R(X, Y)Z, W \rangle &= -B(X, Y)\langle JZ, W \rangle - A(X, Y)\langle IZ, W \rangle.
\end{align*}
\]

for any \( X, Y, Z, W \) tangent to \( M \), where \( A, B, C \) denote certain local 2-forms over \( M \) associated to the local basis \( \{I, J, K\} \).

Henceforth, we suppose \( M \) is a 4\( n \)-dimensional quaternion Kaehler manifold isometrically immersed in \( E^{4n+d} \) \((d < n)\). We denote by \( \sigma \) the second fundamental form of that immersion, and \( T_p M \) (resp. \( T_p^\perp M \)) will denote the tangent space (resp. the normal space) at a point \( p \in M \). Let \( W_p = T_p^\perp M \oplus T_p^\perp M \oplus T_p^\perp M \oplus T_p^\perp M \) be the direct sum of four copies of \( T_p^\perp M \). We consider the indefinite inner product on \( W_p \).

\[
\begin{align*}
\langle \xi_1 \oplus \xi_2 \oplus \xi_3 \oplus \xi_4, \eta_1 \oplus \eta_2 \oplus \eta_3 \oplus \eta_4 \rangle &= 3\langle \xi_1, \eta_1 \rangle - \langle \xi_2, \eta_2 \rangle + \langle \xi_3, \eta_3 \rangle + \langle \xi_4, \eta_4 \rangle.
\end{align*}
\]

and define a bilinear map \( \beta: T_p M \times T_p M \to W_p \) by

\[
\beta(x, y) = \sigma(x, y) \oplus \sigma(x, Iy) \oplus \sigma(x, Jy) \oplus \sigma(x, Ky).
\]

By using (2.1) and the Gauss equation, we have

\[
\langle \langle \beta(x, z), \beta(y, w) \rangle \rangle - \langle \langle \beta(x, w), \beta(y, z) \rangle \rangle = T(x, y, z, w).
\]

where

\[
T(x, y, z, w) = -2\{A(x, y)\langle Iz, w \rangle + B(x, y)\langle Jz, w \rangle + C(x, y)\langle Kz, w \rangle\}.
\]

A priori, \( \beta \) is not flat with respect to \( \langle ., . \rangle \) in the sense of \cite{5}.

For each fixed vector \( x \in T_p M \), \( \beta(x)(y) = \beta(x, y) \) defines a linear map \( \beta(x): T_p M \to W_p \). A vector \( x_0 \) is said to be left regular if

\[
\dim \beta(x_0)(T_p M) = \max \{\dim \beta(x)(T_p M) / x \in T_p M\} = q.
\]

Let

\[
N(\beta, x) = \{ n \in T_p M / \beta(x, n) = 0 \}
\]

be the kernel of \( \beta(x) \).

**Lemma.** Let \( x_0 \) be a left regular vector and \( N_p = N(\beta, x_0) \). Then:

(i) \( N_p \) is a quaternionic subspace of \( T_p M \), i.e., \( I(N_p) = J(N_p) = K(N_p) = N_p \);

(ii) \( \dim N_p \geq 4(n - d) \);

(iii) \( \sigma(n, n') = 0 \) for all \( n, n' \in N_p \).
Proof. Since the proofs of (i) and (ii) are very easy, we will prove (iii).

By using an argument similar to [2, Lemma, (i)], we see that \( \beta(x, n) \in \beta(x_0)(T_pM) \) for all \((x, n) \in T_pM \times N_p\).

Let \( W_1 \) be the subspace generated by \( \beta(T_pM, N_p) \) in \( W_p \). We choose any two vectors \((y_i, n_i) \in T_pM \times N_p, i = 1, 2\). Then there exists \( v_1 \in T_pM \) such that \( \beta(x_0, v_1) = \beta(y_1, n_1) \). Let \( v'_1 \) (resp. \( v''_1 \)) be the \( N_p\)-component (resp. the \( N_p^\perp\)-component, where \( N_p^\perp \) denotes the orthogonal complementary subspace of \( N_p \) in \( T_pM \)) of \( v_1 \). Therefore \( \beta(x_0, v_1) = \beta(x_0, v'_1) \). From (2.4) we have

\[
(2.5) \quad \langle \langle \beta(y_1, n_1), \beta(y_2, n_2) \rangle \rangle = \langle \langle \beta(x_0, v'_1), \beta(y_2, n_2) \rangle \rangle = T(x_0, y_2, v''_1, n_2) = 0
\]

because \( \langle v''_1, In_2 \rangle = \langle v''_1, Jn_2 \rangle = \langle v''_1, Kn_2 \rangle = 0 \). Therefore, \( W_1 \) consists entirely of null vectors.

Let \( \{ \xi_i \oplus \eta_i \oplus v_i \oplus \mu_i, 1 \leq i \leq s \} \) be a basis of \( W_1 \) \((s = \dim W_1)\). Then \( \{ \xi_i, \ 1 \leq i \leq s \} \) is a family of linearly independent vectors in \( T_p^\perp M \), otherwise there exists a linear combination \( \sum_{i=1}^s a_i \xi_i = 0 \) with some \( a_i \neq 0 \), so

\[
(2.6) \quad \sum_{i=1}^s a_i (\xi_i \oplus \eta_i \oplus v_i \oplus \mu_i) = 0 \oplus \eta \oplus v \oplus \mu,
\]

which combined with (2.5) implies \( \eta = v = \mu = 0 \), therefore (2.6) gives a nontrivial linear combination of \( \{ \xi_i \oplus \eta_i \oplus v_i \oplus \mu_i, 1 \leq i \leq s \} \).

We consider the positive definite inner product

\[
(2.7) \quad g(\lambda_1 \oplus \lambda_2 \oplus \lambda_3, \rho_1 \oplus \rho_2 \oplus \rho_3) = \frac{1}{2} \{ \langle \lambda_1, \rho_1 \rangle + \langle \lambda_2, \rho_2 \rangle + \langle \lambda_3, \rho_3 \rangle \}
\]

on \( H_p = T_p^\perp M \oplus T_p^\perp M \oplus T_p^\perp M \).

Since we may suppose \( \{ \xi_i, 1 \leq i \leq s \} \) is a family of orthonormal vectors, by using (2.5) we get

\[
\langle \xi_i, \xi_j \rangle - g(\eta_i \oplus v_i \oplus \mu_i, \eta_j \oplus v_j \oplus \mu_j) = \frac{1}{2} \langle \langle \xi_i \oplus \eta_i \oplus v_i \oplus \mu_i, \xi_j \oplus \eta_j \oplus v_j \oplus \mu_j \rangle \rangle = 0,
\]

which implies \( \{ \eta_i \oplus v_i \oplus \mu_i, 1 \leq i \leq s \} \) is a family of orthonormal vectors in \( (H_p, g) \). Consequently, there exists an orthonormal transformation \( F: T_p^\perp M \to H_p \) such that \( F(\xi_i) = \eta_i \oplus v_i \oplus \mu_i, 1 \leq i \leq s \). In particular, if \((x, n) \in T_pM \times N_p\), then \( \beta(x, n) \in W_1 \), therefore,

\[
(2.8) \quad F(\sigma(x, n)) = \sigma(x, In) \oplus \sigma(x, Jn) \oplus \sigma(x, Kn).
\]

From (i) we know \( In, Jn, Kn \in N_p \) for all \( n \in N_p \), so

\[
\sigma(In, In) \oplus \sigma(In, Jn) \oplus \sigma(In, Kn) = F(\sigma(In, n)) = F(\sigma(n, In)) = \sigma(n, I^2n) \oplus \sigma(n, JIn) \oplus \sigma(n, KIn),
\]

which implies \( \sigma(In, In) = -\sigma(n, n) \) and, similarly, \( \sigma(Jn, Jn) = \sigma(Kn, Kn) = -\sigma(n, n) \). Now clearly, \( \sigma(n, n) = 0 \) for all \( n \in N_p \), which proves (iii).
3. Proof of Main Theorem. We may suppose \( n \geq 2 \) and \( M \) is an Einstein space. According to [6] we have

\[
A(x, y) = -\frac{\tau}{n+2} \langle Ix, y \rangle, \quad B(x, y) = -\frac{\tau}{n+2} \langle Jx, y \rangle, \\
C(x, y) = -\frac{\tau}{n+2} \langle Kx, y \rangle
\]

for any two vectors \( x, y \in T_pM \), where \( 4n \tau \) is the scalar curvature of \( M \).

Let \( n \) be any unit vector in \( N_p \). Then from the Lemma, Gauss equation and (2.1), we get \( A(n, In) = 0 \) so \( \tau = 0 \). This proves \( M \) is Ricci flat.

If \( S \) denotes the Ricci tensor of \( M \), then for any \( n \in N_p \), we have

\[
0 = S(n, n) = -\sum_{i=1}^{4n} \Vert \sigma(n, e_i) \Vert^2,
\]

where \( \{e_i, 1 \leq i \leq 4n\} \) is an orthonormal basis of \( T_pM \). Therefore, we get

\[
\sigma(n, x) = 0 \quad \text{for all} \; n \in N_p \; \text{and} \; x \in T_pM.
\]

On the other hand, let \( RN_p \) be the relative nullity space at \( p \in M \) and \( v_p = \dim RN_p \). From (3.2), \( N_p \subseteq RN_p \) so \( v_p \geq 4(n - d) \). Let \( v_0 = \min \{v_q/q \in M \} \) and \( G = \{q \in M/v_q = v_0 \} \). As is well known, \( G \) is an open subset of \( M \) on which \( RN_p \) (\( p \in G \)) defines an involutive distribution whose leaves are complete \( v_0 \)-planes.

Let \( L \) be the leaf through \( p \in G \). Then since \( RN_p \) contains a quaternionic subspace of dimension \( 4m \geq 4(n - d) \), \( L \) contains \( Q^m \), which is immersed by \( \phi \) identically onto \( E^{4m} \). Since \( M \) is Ricci flat, from [1, Theorem 2], we see that \( M \) is a Riemannian product of \( Q^m \) and a \( 4(n - m) \)-dimensional Riemannian manifold \( P \).

From [3] \( V \) is locally parallelizable, i.e., at each coordinate neighborhood \( U \) there exists local basis \( \{I, J, K\} \) of \( V \) satisfying \( \nabla I = \nabla J = \nabla K = 0 \). Hence \( P \) is a Ricci flat quaternion Kaehlerian manifold with the induced quaternionic structure.

Since the second fundamental form of \( \phi \) satisfies \( \sigma(x, y) = 0 \) for \( x \) tangent to \( Q^m \) and \( y \) tangent to \( P \), [4] implies \( \phi \) splits into a product of the identity map of \( Q^m \) onto \( E^{4m} \) and an isometric immersion of \( P \) into \( E^{4(n-m)+d} \).

In particular, when \( d = 1 \), we get the following characterization of \( Q^n \) with its standard quaternionic structure.

**Corollary.** \( Q^n \) (\( n > 1 \)) is the only complete, simply connected quaternion Kaehler manifold which can be isometrically immersed as a hypersurface of \( E^{4n+1} \).

**References**


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