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AN EXTENSION OF SKOROHOD'S ALMOST SURE REPRESENTATION THEOREM

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ABSTRACT. Skorohod discovered that if a sequence Q_n of countably additive probabilities on a Polish space converges in the weak star topology, then, on a standard probability space, there are Q_n-distributed f_n which converge almost surely. This note strengthens Skorohod's result by associating, with each probability Q on a Polish space, a random variable f_Q on a fixed standard probability space so that for each Q, (a) f_Q has distribution Q and (b) with probability 1, f_P is continuous at $P = Q$.

Let U, λ be any standard probability space, e.g., the unit interval with Lebesgue measure, let X be any Polish space, and denote by $M(X)$ the space of all probability measures on (the Borel sets of) X with the weak star topology. A Borel function ρ: $M(X) \times U \to X$ is a *representation of* $M(X)$ if for every $m \in M(X)$ the function ρ_m: $U \to X$, defined by $\rho_m(u) = \rho(m, u)$, has distribution m (with respect to λ on U). A representation ρ is *continuous at* m if, except for a set of u of λ-measure zero, $\rho(\cdot, u)$ is continuous at m.

THEOREM. *$M(X)$ has a representation ρ that is everywhere continuous.*

The theorem extends a result of Skorohod **[1956]**, who showed that for each sequence $m_n \to m$ there is a sequence r_n of random variables on U, λ such that r_n has distribution m_n and converges almost surely (necessarily to an r with distribution m).

The theorem is easily verified for the special case $X = \mathbf{R}$, with (U, λ) the unit interval with Lebesgue measure and ρ_m the inverse of the c.d.f. of m. (For, unless ρ_m is discontinuous at u, $\rho(m', u)$ is continuous at $m' = m$.) Denote this special continuous representation by ρ^*.

We shall take $U = A \times V$, where $A = [1, 2]$ and $V = [0, 1]$, and take λ = Lebesgue measure on U. Let D be a countable dense subset of X and let $G_1, G_2, \ldots$ be an enumeration of all open spheres with centers in D and with positive rational radii. To define the representation $\rho = \rho(m, a, v)$, fix $a \in A$ and denote by H_n the open sphere with center = center of G_n and with radius equal to the product of a with the radius of G_n, so that the sequence $H_1, H_2, \ldots$ also determines the topology of X. Map X into $\mathbf{R}$ by c: $c(x) = \Sigma H_n(x)/3^n$ (using de Finetti's convention, $H_n(x) = 1$ or 0 according as $x \in H_n$ or not), and let $\mu = J(m)$ be the distribution of c when x has distribution m. Since the H_n separate points, c is easily verified to be injective and to

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possess an inverse $h: c(X) \to X$. Now define ρ, thus:

$$\rho(m, a, v) = h_a(\rho^*(J_a(m), v)), \tag{1}$$

where the subscript a indicates the dependence of h and J on a.

Now for a sketch of the proof that ρ is an everywhere continuous representation of $M(X)$.

(A) *ρ is a representation* of $M(X)$. Since each of its component functions h, ρ^* and J is demonstrably a Borel function of its two arguments, ρ is Borel. For each fixed a, the function r_a: $M(X) \times V \to X$ defined by $r_a(m, v) = \rho(m, a, v)$ is a representation of $M(X)$ on the standard space V, Lebesgue measure, for the function $\rho^*(J_a(m), \cdot)$ has distribution $J_a(m)$, so that $h_a(\rho^*(J_a(m), \cdot))$ has distribution m. Since the conditional distribution of ρ_m given a is m for each a, its unconditional distribution is m.

(B) *h is continuous* on $c(X)$. For $h^{-1}(H)$, namely $c(H)$, is open in $c(X)$ for H equal to any H_n and, hence, for H equal to any open subset of X.

(C) *The representation r_a (defined in A) is continuous at each m for which the boundaries of the spheres H_n have m-measure zero.* For, say, $m_k \to m$. To verify that J is continuous is to verify that $m_k(S) \to m(S)$ for S in the field determined by the H_n. The class of sets whose boundary has m-measure zero is always a field, so that each S has a boundary of m-measure zero, and $m_k \to m$ implies $m_k(S) \to m(S)$. Since J is continuous, $\rho^*(J(m'), v)$ is continuous at $m' = m$ for all v other than those where $\rho^*(m, \cdot)$ is discontinuous. Finally, for all such v, continuity of h gives continuity of $h(\rho^*(J(m'), v))$ at $m' = m$; that is, $r_a(m', v)$ is continuous at $m' = m$ for all except a countable set of v.

(D) *ρ is continuous at each m.* Say $m' \to m$. From C, the probability, given a, that $\rho_{m'} \to \rho_m$ is 1 if the boundaries of all H_n have m-measure zero. Since there are only countably many a for which the boundary of any H_n has positive m-measure, the event $\rho_{m'} \to \rho_m$ has conditional probability 1 for all except countably many a's, so has unconditional probability 1.

COROLLARY. *If $X(t)$ is an X-valued stochastic process that is continuous in distribution at every t, there is a stochastic process $Y(t)$ such that, for every t,* (a) *$Y(t)$ has the same distribution as $X(t)$, and* (b) *with probability* 1, *Y is continuous at t.*

PROOF. Take $Y(t) = \rho_{m(t)}$, where $m(t)$ is the distribution of $X(t)$.

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