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# SELECTORS FOR BOREL SETS WITH LARGE SECTIONS

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ABSTRACT. We prove a result asserting the existence of a Borel selector for a Borel set in the product of two Polish spaces. This subsumes a number of results about Borel selectors for Borel sets having large sections.

**1. Introduction.** The principal selection theorems for Borel sets with large sections can be stated as follows. If $X$ is analytic, $Y$ a Polish space, $B$ a Borel subset of $X \times Y$ such that every vertical section $B_x$ is of positive measure or nonmeager, then $B$ admits a Borel measurable selector, i.e., there exists a Borel measurable function $f$: $X \to Y$ such that $(x, f(x)) \in B$ for each $x \in X$. The measure result was proved by Blackwell and Ryll-Nardzewski [**1**], while the category result has been observed by several authors (see, for example, [**5** and **6**]). A selection theorem of Burgess [**2**], when specialized to the situation above, extends the category result and subsumes a result of Srivastava [**8**] concerning Borel selectors for the case when the sections $B_x$ are $G_\delta$ sets (see also [**4**] in this connection). A precise version of Burgess' result will be given in §4.

The aim of the present article is to show that the results described above are really instances of a single theorem. Moreover, the proof we give here is elementary in nature, using nothing more than the well-known characterization of a Borel subset of a Polish space as a one-one, continuous, bimeasurable image of a closed subset of the space of irrationals (see, for instance, [**3**]). In particular, the separation principle for analytic sets is avoided. Lastly, the proof works for $X$, an arbitrary separable metric space (Srivatsa [**7**] observed that the results mentioned in the previous paragraph hold for separable metric $X$, not just for analytic $X$).

The situation obtaining for Borel sets with large sections should be contrasted with that for Borel sets with small sections, i.e., when all vertical sections are countable or compact or $\sigma$-compact. It is known that such Borel sets admit Borel selectors when the horizontal axis $X$ is analytic [**5**]. The proofs of these results, however, involve in an essential manner the separation principle for analytic sets (indeed, the separation principle for a sequence of analytic sets). Furthermore, the results are not true for arbitrary separable metric $X$. (I am indebted to J. P. Burgess for the last remark.)

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The main result of this paper is as follows.

THEOREM. *Let $X$ be a separable metric space and $Y$ a Polish space. For each $x \in X$, let $\mathcal{I}_x$ be a $\sigma$-ideal of subsets of the Borel $\sigma$-field of $Y$ such that for each Borel set $C$ in $X \times Y$, the set $\{x \in X: C_x \notin \mathcal{I}_x\}$ is Borel in $X$, where $C_x = \{y \in Y: (x, y) \in C\}$.*

*Let $B$ be a Borel set in $X \times Y$ such that $B_x \notin \mathcal{I}_x$ for each $x \in X$. Then there is a Borel measurable function $f: X \to Y$ such that $(x, f(x)) \in B$ for each $x \in X$.*

The proof of the theorem will be given in §3. §2 explains the notation to be used. §4 will discuss special cases of our result.

**2. Notation.** We denote the set of natural numbers by $\omega$. The letters $k$, $m$, $n$, with or without primes, will stand for natural numbers. The set of finite sequences of natural numbers is denoted by $\omega^{<\omega}$. Elements of $\omega^{<\omega}$ will be denoted by $s$, $t$, with or without primes. The letter $e$ will denote the empty sequence in $\omega^{<\omega}$. If $s \in \omega^{<\omega}$, $\mathrm{lh}(s)$ denotes the length of $s$. If $s \in \omega^{<\omega}$ and $m \in \omega$, $sm$ denotes the catenation of the sequence $s$ followed by the sequence $\langle m \rangle$.

The set of infinite sequences of natural numbers will be denoted by $\omega^\omega$. Elements of $\omega^\omega$ will be denoted by $\alpha, \beta$. For $\alpha \in \omega^\omega$ and $n \in \omega$, $\alpha(n)$ stands for the $n$th coordinate of $\alpha$ and $\alpha \restriction n$ will denote the finite sequence $\langle \alpha(0), \alpha(1), \ldots, \alpha(n-1) \rangle$. For $s \in \omega^{<\omega}$ and $\alpha \in \omega^\omega$, write $s \subseteq \alpha$ if $\alpha \restriction \mathrm{lh}(s) = s$. We denote the set $\{\alpha \in \omega^\omega: s \subseteq \alpha\}$ by $N(s)$ for $s \in \omega^{<\omega}$. The sets $N(s)$, $s \in \omega^{<\omega}$, form a base for a topology on $\omega^\omega$. Endowed with this topology, $\omega^\omega$ becomes a Polish space which is homeomorphic to the space of irrationals.

**3. Proof of theorem.** Fix a complete metric $d$ on $Y$ such that $d -$ diameter$(Y) < 1$. Choose a system $\{V(s): s < \omega^{<\omega}\}$ of subsets of $Y$ satisfying:

(i) $V(e) = Y$.

(ii) $V(s)$ is nonempty and open in $Y$.

(iii) $V(s) = \bigcup_{m \geqslant 0} V(sm)$.

(iv) $\mathrm{cl}(V(sm)) \subseteq V(s)$, where cl denotes closure in $Y$.

(v) $d -$ diameter$(V(s)) < 2^{-\mathrm{lh}(s)}$.

Let $\tilde{X}$ be a completion of $X$, so $\tilde{X}$ is a Polish space. Since $B$ is Borel in $X \times Y$, we can find a Borel set $\tilde{B}$ in $\tilde{X} \times Y$ such that $B = \tilde{B} \cap (X \times Y)$. Fix a closed subset $D$ of $\omega^\omega$ and a one-one, continuous function $g$ on $D$ onto $\tilde{B}$ such that $g$ takes Borel subsets of $D$ to Borel sets in $\tilde{X} \times Y$.

Next we define a system $\{T(s, t): s, t \in \omega^{<\omega} \,\&\, \mathrm{lh}(s) = \mathrm{lh}(t)\}$ of subsets of $X$ satisfying:

(a) $T(s, t)$ is a Borel set in $X$,

(b) $T(e, e) = X$,

(c) $T(s, t) \cap T(s', t') = \varnothing$ if $(s, t) \neq (s', t') \,\&\, \mathrm{lh}(s) = \mathrm{lh}(s')$,

(d) $T(s, t) = \bigcup_{m \geqslant 0} \bigcup_{n \geqslant 0} T(sm, tn)$,

(e) $T(s, t) \subseteq (g(D \cap N(s)) \cap (X \times Y))^{*V(t)}$,

where $(g(D \cap N(s)) \cap (X \times Y))^{*V(t)} = \{x \in X: (g(D \cap N(s)) \cap (X \times V(t)))_x \notin \mathcal{I}_x\}$. The definition of $T(s, t)$ is by induction on $\mathrm{lh}(s)$. Start the definition by setting $T(e, e) = X$. Assume that $T(s', t')$ has been defined for all $s'$, $t^1 \in \omega^{<\omega}$ such that

$\mathrm{lh}(s') = \mathrm{lh}(t') \leqslant k$. Fix $s, t \in \omega^{<\omega}$ such that $\mathrm{lh}(s) = \mathrm{lh}(t) = k$. Set

$$T'(m, n) = (g(D \cap N(sm)) \cap (X \times Y))^{*V(tn)} \cap T(s, t), \qquad m, n \in \omega.$$

Since $g(D \cap N(sm))$ is Borel in $\tilde{X} \times Y$, the set $g(D \cap N(sm)) \cap (X \times V(tn))$ is Borel in $X \times Y$. So, by hypothesis, $(g(D \cap N(sm)) \cap (X \times Y))^{*V(tn)}$ is Borel in $X$, hence $T'(m, n)$ is Borel in $X$. We claim that $T(s, t) = \bigcup_{m \geqslant 0} \bigcup_{n \geqslant 0} T'(m, n)$. To see this, let $x \in T(s, t)$. Note that

$$(g(D \cap N(s)))_x \cap V(t) = \bigcup_{m \geqslant 0} \bigcup_{n \geqslant 0} [(g(D \cap N(sm)))_x \cap V(tn)].$$

Since $x \in T(s, t)$, the induction hypothesis and clause (e) above imply that

$$(g(D \cap N(s)))_x \cap V(t) \notin \mathcal{I}_x.$$

Since $\mathcal{I}_x$ is a $\sigma$-ideal, it follows that there exist $m, n \in \omega$ such that $(g(D \cap N(sm)))_x \cap V(tn) \notin \mathcal{I}_x$, so $x \in T'(m, n)$. Next, disjointify the sets $T'(m, n)$: get Borel sets $T''(m, n)$ in $X$ such that $T''(m, n) \subseteq T'(m, n)$, $T''(m, n) \cap T''(m', n') = \varnothing$ if $(m, n) \neq (m', n')$, and $\bigcup_{m \geqslant 0} \bigcup_{n \geqslant 0} T''(m, n) = \bigcup_{m \geqslant 0} \bigcup_{n \geqslant 0} T'(m, n)$. Finally, set $T(sm, tn) = T''(m, n)$. The sets so obtained satisfy clauses (a)–(e).

Define

$$G = \bigcap_{k \geqslant 0} \bigcup [T(s, t) \times \mathrm{cl}(V(t))],$$

where the inner union is over all ordered pairs $(s, t) \in \omega^{<\omega} \times \omega^{<\omega}$ such that $\mathrm{lh}(s) = \mathrm{lh}(t) = k$.

We make two observations about $G$: (1) $G_x$ is a singleton for each $x \in X$, and (2) $G \subseteq B$. To see (1), let $x \in X$. Then there is a unique $(\alpha, \beta) \in \omega^\omega \times \omega^\omega$ such that $x \in T(\alpha \upharpoonright k, \beta \upharpoonright k)$ for all $k \geqslant 0$. This follows from clauses (b)–(d) in the definition of the system $\{T(s, t)\}$. So $G_x = \bigcap_{k \geqslant 0} \mathrm{cl}(V(\beta \upharpoonright k))$, which is a singleton set by (iv), (v) and the fact that $Y$ is Polish. For (2), let $(x, y) \in G$. Find $(\alpha, \beta) \in \omega^\omega \times \omega^\omega$ such that $x \in T(\alpha \upharpoonright k, \beta \upharpoonright k)$ for all $k \geqslant 0$. Hence $\{y\} = \bigcap_{k \geqslant 0} \mathrm{cl}(V(\beta \upharpoonright k))$. Since $x \in T(\alpha \upharpoonright k, \beta \upharpoonright k)$ for each $k$, it follows by (e) that $(g(D \cap N(\alpha \upharpoonright k)))_x \cap V(\beta \upharpoonright k) \neq \varnothing$ for each $k \geqslant 0$. Consequently, $y \in \bigcap_{k \geqslant 0}(\mathrm{cl}(g(D \cap N(\alpha \upharpoonright k))))_x$, so $(x, y) \in \bigcap_{k \geqslant 0} \mathrm{cl}(g(D \cap N(\alpha \upharpoonright k))) = \{g(\alpha)\}$, using continuity of $g$ and closedness of $D$. Consequently, $(x, y) \in B$.

Let $f\colon X \to Y$ be the function whose graph is $G$. Since $G \subseteq B$, it follows that $(x, f(x)) \in B$ for every $x \in x$. To see that $f$ is Borel measurable, observe that for any open set $W$ in $Y$,

$$f^{-1}(W) = \bigcup \{T(s, t)\colon \mathrm{lh}(s) = \mathrm{lh}(t) \,\&\, \mathrm{cl}(V(t)) \subseteq W\},$$

so that $f^{-1}(W)$ is Borel in $X$. This completes the proof.

**4. Special cases.** In this final section, we deduce the results mentioned in the Introduction from our result.

Let $X$ be a separable metric space, $Y$ a Polish space and $B$ a Borel set in $X \times Y$.

1°. To consider the measure result first, let $Q(x, E)$, $x \in X$, $E$ a Borel set in $Y$, be a Borel measurable transition function such that $Q(x, B_x) > 0$ for each $x \in X$. In our theorem, we take $\mathcal{I}_x = \{E \subseteq Y\colon E \text{ is Borel } \& \ Q(x, E) = 0\}$, $x \in X$. It is a standard fact of measure theory that the $\sigma$-ideals $\mathcal{J}_x$, $x \in X$, satisfy the condition of our theorem, so that the Blackwell-Ryll-Nardzewski result falls out of ours.

2°. For the category case, assume that $B_x$ is nonmeager for each $x \in X$. We take $\mathcal{J}_x = \{E \subseteq Y\colon E \text{ is Borel and meager}\}$, $x \in X$. That the $\mathcal{J}_x$, $x \in X$, satisfy the condition of our theorem is a result of P. S. Novikov and, independently, of Vaught **[9]**. The category result is then an immediate consequence of our result.

3°. In Burgess' result, the assumptions on the Borel set $B$ are as follows: (a) the multifunction $x \to B_x$ is Borel measurable, i.e., the set $\{x \in X\colon B_x \cap V \neq \varnothing\}$ is Borel in $X$ for each open set $V$ in $Y$, and (b) $B_x$ is nonmeager in $\mathrm{cl}(B_x)$ for each $x \in X$. We set

$$\mathcal{J}_x = \{E \subseteq \mathrm{cl}(B_x)\colon E \text{ is Borel and meager in } \mathrm{cl}(B_x)\}, \qquad x \in X.$$

A very easy modification of Vaught's proof of the fact mentioned in 2° now shows that the $\sigma$-ideals $\mathcal{J}_x$, $x \in X$, satisfy the condition of our theorem. Our theorem then implies that $B$ admits a Borel measurable selector, which is Burgess' result.

Srivatsa [7] has an ingenious way of deducing Burgess' result from the category result in 2°. Finally, routine arguments show that the results in 1°–3° hold for an arbitrary measurable space $(X, \mathcal{A})$, since they hold for a separable metric space (see [7]).

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