NOTE ON ROTATION SET

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Abstract. Let \( f \) be an endomorphism of the circle of degree 1 and \( \tilde{f} \) be a lifting of \( f \). We characterize the rotation set \( \rho(f) \) by the set of probability measures on the circle, and prove that if \( \rho_{+}(\tilde{f}) \neq \rho_{-}(\tilde{f}) \), the upper (lower) endpoint of \( \rho(\tilde{f}) \), is irrational, then \( \rho_{+}(R_{\theta}\tilde{f}) > \rho_{+}(\tilde{f}) \) for any \( \theta > 0 \), where \( R_{\theta}(x) = x + \theta \). As a corollary, if \( f \) is structurally stable, then both \( \rho_{+}(\tilde{f}) \) and \( \rho_{-}(\tilde{f}) \) are rational.

Newhouse, Palis and Takens [3] have generalized a rotation number for a homeomorphism of the circle to a continuous map of degree 1 and defined a rotation set. Let \( R \) denote the real numbers, \( Z \) the integers, \( N \) the positive integers and \( S = R/Z \) the circle. Let \( \pi: R \to S \) denote the canonical projection. Let \( f: S \to S \) be a given continuous map of degree 1. Choose a lifting \( \tilde{f}: R \to R \), that is a map such that \( \pi \tilde{f} = f \pi \). Liftings exist and are unique up to the addition of an integer. Each lifting satisfies \( \tilde{f}(x + 1) = \tilde{f}(x) + 1 \).

Definition. Given \( x \in R \), define the rotation number

\[
\rho(\tilde{f}, x) = \limsup_{n \to \infty} \frac{1}{n} (\tilde{f}^n(x) - x).
\]

Define the rotation set to be \( \rho(\tilde{f}) = \{\rho(\tilde{f}, x) | x \in R \} \).

Notice that if a different lifting is used, this nearly has the effect of translating the rotation set by an integer.

We recall the following properties of \( \rho(\tilde{f}) \).

1. If \( f = hgh^{-1} \) for an orientation preserving homeomorphism \( h \) of \( S \), then \( \rho(\tilde{f}) = \rho(\tilde{g}) \) for suitable liftings \( \tilde{f} \) and \( \tilde{g} \).

2. \( \rho(\tilde{f}) \) is either one point or a closed interval.

3. For any \( \alpha \in \rho(\tilde{f}) \), there exists \( x \in R \) such that \( \lim_{n \to \infty}(\tilde{f}^n(x) - x)/n = \alpha \).

(1) is trivial from the definition. See [2] and [3] for (2), (3) and other elementary properties of \( \rho(\tilde{f}) \). By (2) we may denote the upper and lower endpoints of \( \rho(\tilde{f}) \) by \( \rho_{+}(\tilde{f}) \) and \( \rho_{-}(\tilde{f}) \) respectively.

The aim of this paper is to prove the following

**Theorem 1.** Let \( M \) be the set of probability measures on \( S \) invariant with respect to \( f \). Let \( \varphi = \tilde{f} - \text{Id}: S \to R \) where \( \text{Id}: R \to R \) denotes identity. Then \( \rho(\tilde{f}) = \{\mu(\varphi) | \mu \in M \} \).

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Notes. (i) The set of probability measures on \( S \) is regarded as the set of positive linear functionals \( \mu \) on \( C(S) \) such that \( \mu(1) = 1 \).

(ii) \( S \) is considered to be \([0, 1]\) with 0 and 1 identified, and \( \varphi(x) = \hat{f}(x) - x \) for \( x \in [0, 1] \).

Theorem 2. If \( \rho, (\hat{f}) \) (\( \rho, (\hat{f}) \)) is irrational and \( \theta > 0 \), then \( \rho, (R_\theta \hat{f}) > \rho, (\hat{f}) \) (\( \rho, (R_\theta \hat{f}) > \rho, (\hat{f}) \)) where \( R_\theta : R \to R \) is defined as \( R_\theta(x) = x + \theta \).

These two theorems are generalizations of properties given in Herman [1] for the case \( f \) is a homomorphism of \( S \).

Considering (1) and Theorem 2, we obtain the following

Corollary. If \( f \) is structurally stable, then both \( \rho, (\hat{f}) \) and \( \rho, (\hat{f}) \) are rational.

Proof of Theorem 1. By (3) above, for any \( \alpha \in \rho(f) \), there exists \( x \in S \) satisfying

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{\ell=0}^{n-1} \varphi(f'(x)) = \lim_{n \to \infty} \frac{1}{n} (\hat{f}^n(x) - x) = \alpha
\]

where \( \varphi = \hat{f} - \text{Id} \).

Define \( \mu_n : C(S) \to \mathbb{R} \) by \( \mu_n(g) = n^{-1} \sum_{\ell=0}^{n-1} g(f'(x)) \) for \( g \in C(S) \) using \( x \) above. Then \( \mu_n \) is a probability measure on \( S \). Since the set of probability measures is weak* compact, there is a subsequence \( \{\mu_{n_k}\} \) of \( \{\mu_n\} \) which converges weakly to a probability measure, say \( \mu \). That is, we have \( \mu_{k_n}(g) \to \mu(g) \) for any \( g \in C(S) \).

Taking \( \varphi = f - \text{Id} \) for \( g \), we have \( \mu_{k_n}(\varphi) \to \mu(\varphi) \), while by definition,

\[
\lim_{n \to \infty} \mu_{k_n}(\varphi) = \lim_{n \to \infty} \frac{1}{k_n} \sum_{i=0}^{k_n-1} \varphi(f'(x)) = \alpha.
\]

Therefore, we have \( \mu(\varphi) = \alpha \). Since \( \lim_{n \to \infty} \mu_{k_n}(g) = \mu(g) \), \( \mu \) is invariant with respect to \( f \).

Consequently \( \rho(\hat{f}) \subset \{\mu(\varphi) | \mu \in M\} \). On the other hand, for any rational \( p/q < \rho, (f) \), we have \( \hat{f}^q(x) - x - p > 0 \) for any \( x \in S \). Thus \( \mu(\hat{f}^q - \text{Id} - q \cdot p/q) > 0 \) for any \( \mu \in M \), while

\[
\mu(\hat{f}^q - \text{Id} - q\mu(\varphi)) = \mu\left(\sum_{i=0}^{q-1} (\hat{f} - \text{Id}) \circ f^i\right) - q\mu(\varphi) = 0.
\]

Therefore \( \mu(\varphi) > p/q \).

By the same reasoning, we have \( \mu(\varphi) < p/q \) for any rational \( p/q > \rho, (f) \). Since \( \rho(\hat{f}) \) is a closed set and \( \rho(\hat{f}) \subset \{\mu(\varphi) | \mu \in M\} \), we have \( \rho(\hat{f}) = \{\mu(\varphi) | \mu \in M\} \).

We need two lemmas to prove Theorem 2.

Lemma 1. For any irrational number \( \alpha \) there exists a monotone decreasing sequence \( \{p_n/q_n\} \) (and a monotone increasing sequence \( \{p'_n/q'_n\} \)) of rationals converging to \( \alpha \) and satisfying \( p_n/q_n - \alpha < 1/q_n^2 \) \( (\alpha - p'_n/q'_n < 1/q_n^2) \).

As this is a well-known fact in arithmetic, we do not give the proof.
Lemma 2. Let $\theta > 0$. For any $k \in \mathbb{N}$ and any $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that $y \leq x$ and $(R_\theta \hat{f})^k(y) \geq \hat{f}^k(x) + \theta$.

Proof. We prove this by induction. For $k = 1$ it is trivial. Assume the lemma is true for $k$, then we have $y \leq x$ and $(R_\theta \hat{f})^k(y) \geq \hat{f}^k(x) + \theta$. Since $(R_\theta \hat{f})^k(s + n) = (R_\theta \hat{f})^k(s) + n$ for $n \in \mathbb{Z}$, we have $z \leq y$ such that $(R_\theta \hat{f})^k(z) = \hat{f}^k(x)$, then

$$(R_\theta \hat{f})^{k+1}(z) = \hat{f}((R_\theta \hat{f})^k(z)) + \theta$$

$= \hat{f}(\hat{f}^k(x)) + \theta = \hat{f}^{k+1}(x) + \theta$$

and $z \leq x$, completing the induction. □

Proof of Theorem 2. We prove the theorem for $\rho_+ (\hat{f})$ because the $\rho_- (\hat{f})$ case is similar. By Lemma 1, we may choose a sequence $\{p_n/q_n\}$ of rationals such that $\{p_n/q_n\} \uparrow \alpha$ and $p_n/q_n - 1/q_n^2 < \alpha$. Since $\rho_+ (\hat{f}) = \alpha$, we have $\hat{f}^{q_n}(x) - x < p_n$ for any $x \in \mathbb{R}$. Suppose that there exists $\theta > 0$ such that $\hat{f}^{q_n}(x) - x < p_n - \theta$ for any $n \in \mathbb{N}$ and any $x \in \mathbb{R}$. Take $q_n$ large enough to satisfy $q_n \theta > 1$, then we have

$$\hat{f}^{q_n}(x) - x = \sum_{i=0}^{q_n-1} \{\hat{f}^{i}(\hat{f}^{q_n}(x)) - \hat{f}^{i+1}(x)\} < q_n(p_n - \theta) < q_n p_n - 1.$$  

Thus, we have $\rho_+ (\hat{f}) < (q_n p_n - 1)/q_n^2 < \alpha$, contradicting $\rho_+ (\hat{f}) = \alpha$. Therefore, for any $\theta > 0$, there exist $n \in \mathbb{N}$ and $x \in \mathbb{R}$ such that $\hat{f}^{q_n}(x) - x \geq p_n - \theta$. On the other hand, by Lemma 2, there exists $y \leq x$ satisfying $(R_\theta \hat{f})^{q_n}(y) \geq \hat{f}^{q_n}(x) + \theta$. Thus, we have $(R_\theta \hat{f})^{q_n}(y) - y \geq \hat{f}^{q_n}(x) - x + \theta \geq p_n$. Therefore, $\rho_+ (R_\theta \hat{f}) \geq p_n/q_n > \alpha$, completing the proof. □

References


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