NOTE ON ROTATION SET

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Abstract. Let $f$ be an endomorphism of the circle of degree 1 and $\hat{f}$ be a lifting of $f$. We characterize the rotation set $\rho(f)$ by the set of probability measures on the circle, and prove that if $\rho_+ (\hat{f}) (\rho_-(\hat{f}))$, the upper (lower) endpoint of $\rho(\hat{f})$, is irrational, then $\rho_+ (R_{\theta} \hat{f}) > \rho_+ (\hat{f}) (\rho_-(R_{\theta} \hat{f}) > \rho_-(\hat{f}))$ for any $\theta > 0$, where $R_{\theta}(x) = x + \theta$. As a corollary, if $f$ is structurally stable, then both $\rho_+ (f)$ and $\rho_-(f)$ are rational.

Newhouse, Palis and Takens [3] have generalized a rotation number for a homeomorphism of the circle to a continuous map of degree 1 and defined a rotation set. Let $R$ denote the real numbers, $Z$ the integers, $N$ the positive integers and $S = R/Z$ the circle. Let $\pi: R \to S$ denote the canonical projection. Let $f: S \to S$ be a given continuous map of degree 1. Choose a lifting $\hat{f}: R \to R$, that is a map such that $\pi \hat{f} = f \pi$. Liftings exist and are unique up to the addition of an integer. Each lifting satisfies $\hat{f}(x+1) = \hat{f}(x) + 1$.

Definition. Given $x \in R$, define the rotation number

$$\rho(\hat{f}, x) = \limsup_{n \to \infty} \frac{1}{n} (\hat{f}^n(x) - x).$$

Define the rotation set to be $\rho(\hat{f}) = \{\rho(\hat{f}, x) | x \in R\}$.

Notice that if a different lifting is used, this nearly has the effect of translating the rotation set by an integer.

We recall the following properties of $\rho(\hat{f})$.

(1) If $f = hg^{-1}$ for an orientation preserving homeomorphism $h$ of $S$, then $\rho(\hat{f}) = \rho(\hat{g})$ for suitable liftings $\hat{f}$ and $\hat{g}$.

(2) $\rho(\hat{f})$ is either one point or a closed interval.

(3) For any $\alpha \in \rho(\hat{f})$, there exists $x \in R$ such that $\lim_{n \to \infty} (f^n(x) - x)/n = \alpha$.

(1) is trivial from the definition. See [2] and [3] for (2), (3) and other elementary properties of $\rho(\hat{f})$. By (2) we may denote the upper and lower endpoints of $\rho(\hat{f})$ by $\rho_+ (\hat{f})$ and $\rho_-(\hat{f})$ respectively.

The aim of this paper is to prove the following

Theorem 1. Let $M$ be the set of probability measures on $S$ invariant with respect to $f$. Let $\varphi = \hat{f} - \text{Id}: S \to R$ where $\text{Id}: R \to R$ denotes identity. Then $\rho(\hat{f}) = \{\mu(\varphi) | \mu \in M\}$. 

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Notes. (i) The set of probability measures on \( S \) is regarded as the set of positive linear functionals \( \mu \) on \( C(S) \) such that \( \mu(1) = 1 \).

(ii) \( S \) is considered to be \([0, 1] \) with 0 and 1 identified, and \( \varphi(x) = f(x) - x \) for \( x \in [0, 1] \).

**Theorem 2.** If \( \rho_\theta(f) \) is irrational and \( \theta > 0 \), then \( \rho_\theta(f) > \rho_\theta(f) \) (\( \rho_\theta(f) > \rho_\theta(f) \)) where \( R_\theta : R \rightarrow R \) is defined as \( R_\theta(x) = x + \theta \).

These two theorems are generalizations of properties given in Herman [I] for the case \( f \) is a homomorphism of \( S \).

Considering (i) and Theorem 2, we obtain the following

**Corollary.** If \( f \) is structurally stable, then both \( \rho_\theta(f) \) and \( \rho_\theta(f) \) are rational.

**Proof of Theorem 1.** By (3) above, for any \( \alpha \in \rho(f) \), there exists \( x \in S \) satisfying

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)) = \lim_{n \to \infty} \frac{1}{n} (\tilde{f}^n(x) - x) = \alpha
\]

where \( \varphi = f - \text{Id} \).

Define \( \mu_n : C(S) \rightarrow R \) by \( \mu_n(g) = n^{-1} \sum_{i=0}^{n-1} g(f^i(x)) \) for \( g \in C(S) \) using \( x \) above. Then \( \mu_n \) is a probability measure on \( S \). Since the set of probability measures is weak* compact, there is a subsequence \( \{ \mu_{n_k} \} \) of \( \{ \mu_n \} \) which converges weakly to a probability measure, say \( \mu \). That is, we have \( \mu_{n_k}(g) \rightarrow \mu(g) \) for any \( g \in C(S) \).

Taking \( \varphi = f - \text{Id} \) for \( g \), we have \( \mu_{n_k}(\varphi) = \mu(\varphi) \), while by definition,

\[
\lim_{n \to \infty} \mu_{n_k}(\varphi) = \lim_{n \to \infty} \frac{1}{k_n} \sum_{i=0}^{k_n-1} \varphi(f^i(x)) = \alpha.
\]

Therefore, we have \( \mu(\varphi) = \alpha \). Since \( \lim_{n \to \infty} \mu_{n_k}(g) = \mu(g) \), \( \mu \) is invariant with respect to \( f \).

Consequently \( \rho(f) \subset \{ \mu(\varphi) | \mu \in \mathcal{M} \} \). On the other hand, for any rational \( p/q < \rho_\theta(f) \), we have \( \tilde{f}^q(x) - x - p > 0 \) for any \( x \in S \). Thus \( \mu(\tilde{f}^q - \text{Id} - q \cdot f/q) > 0 \) for any \( \mu \in \mathcal{M} \), while

\[
\mu(f^q - \text{Id} - q\varphi(f)) = \mu \left( \sum_{i=0}^{q-1} (\tilde{f} - \text{Id}) \circ f^i \right) = q\mu(\varphi) = 0.
\]

Therefore \( \mu(\varphi) > p/q \).

By the same reasoning, we have \( \mu(\varphi) < p/q \) for any rational \( p/q > \rho_\theta(f) \). Since \( \rho(f) \) is a closed set and \( \rho(f) \subset \{ \mu(\varphi) | \mu \in \mathcal{M} \} \), we have \( \rho(f) = \{ \mu(\varphi) | \mu \in \mathcal{M} \} \).

\( \Box \)

We need two lemmas to prove Theorem 2.

**Lemma 1.** For any irrational number \( \alpha \) there exists a monotone decreasing sequence \( \{ p_n/q_n \} \) (and a monotone increasing sequence \( \{ p'_n/q'_n \} \)) of rationals converging to \( \alpha \) and satisfying \( p_n/q_n - \alpha < 1/q_n^2 \) (\( \alpha - p'_n/q'_n < 1/q_n^2 \)).

As this is a well-known fact in arithmetic, we do not give the proof.
Lemma 2. Let $\theta > 0$. For any $k \in N$ and any $x \in R$, there exists $y \in R$ such that $y \leq x$ and $(R_{\theta} \hat{f})^k(y) \geq \hat{f}^k(x) + \theta$.

Proof. We prove this by induction. For $k = 1$ it is trivial. Assume the lemma is true for $k$, then we have $y \leq x$ and $(R_{\theta} \hat{f})^k(y) \geq \hat{f}^k(x) + \theta$. Since $(R_{\theta} \hat{f})^k(s + n) = (R_{\theta} \hat{f})^k(s) + n$ for $n \in Z$, we have $z \leq y$ such that $(R_{\theta} \hat{f})^k(z) = \hat{f}^k(x)$, then

$$(R_{\theta} \hat{f})^{k+1}(z) = \hat{f}((R_{\theta} \hat{f})^k(z)) + \theta$$

and $z \leq x$, completing the induction.

Proof of Theorem 2. We prove the theorem for $\rho_+(\hat{f})$ because the $\rho_-(\hat{f})$ case is similar. By Lemma 1, we may choose a sequence $\{p_n/q_n\}$ of rationals such that $\{p_n/q_n\} \downarrow \alpha$ and $p_n/q_n - 1/q_n^2 < \alpha$. Since $\rho_+(\hat{f}) = \alpha$, we have $\hat{f}^{q_n}(x) - x < p_n$ for any $x \in R$. Suppose that there exists $\theta > 0$ such that $\hat{f}^{q_n}(x) - x < p_n - \theta$ for any $n \in N$ and any $x \in R$. Take $q_n$ large enough to satisfy $q_n \theta > 1$, then we have

$$\hat{f}^{q_n}(x) - x = \sum_{i=0}^{q_n-1} \left\{ \hat{f}^{q_n}(\hat{f}^{i}x(x)) - \hat{f}^{i+1}x(x) \right\} < q_n(p_n - \theta) < q_n p_n - 1.$$ 

Thus, we have $\rho_+(\hat{f}) < (q_n p_n - 1)/q_n^2 < \alpha$, contradicting $\rho_+(\hat{f}) = \alpha$. Therefore, for any $\theta > 0$, there exist $n \in N$ and $x \in R$ such that $\hat{f}^{q_n}(x) - x \geq p_n - \theta$. On the other hand, by Lemma 2, there exists $y \leq x$ satisfying $(R_{\theta} \hat{f})^{q_n}(y) \geq \hat{f}^{q_n}(x) + \theta$. Thus, we have $(R_{\theta} \hat{f})^{q_n}(y) - y \geq \hat{f}^{q_n}(x) - x + \theta \geq p_n$. Therefore, $\rho_+(R_{\theta} \hat{f}) \geq p_n/q_n > \alpha$, completing the proof.

References

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