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NOTE ON ROTATION SET

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ABSTRACT. Let f be an endomorphism of the circle of degree 1 and $\bar{f}$ be a lifting of f. We characterize the rotation set $\rho(\bar{f})$ by the set of probability measures on the circle, and prove that if $\rho_+(\bar{f})$ ($\rho_-(\bar{f})$), the upper (lower) endpoint of $\rho(\bar{f})$, is irrational, then $\rho_+(R_\theta \bar{f}) > \rho_+(\bar{f})$ ($\rho_-(R_\theta \bar{f}) > \rho_-(\bar{f})$) for any $\theta > 0$, where $R_\theta(x) = x + \theta$. As a corollary, if f is structurally stable, then both $\rho_+(\bar{f})$ and $\rho_-(\bar{f})$ are rational.

Newhouse, Palis and Takens [3] have generalized a rotation number for a homeomorphism of the circle to a continuous map of degree 1 and defined a rotation set. Let R denote the real numbers, Z the integers, N the positive integers and $S = R/Z$ the circle. Let π: $R \to S$ denote the canonical projection. Let f: $S \to S$ be a given continuous map of degree 1. Choose a lifting $\bar{f}$: $R \to R$, that is a map such that $\pi\bar{f} = f\pi$. Liftings exist and are unique up to the addition of an integer. Each lifting satisfies $\bar{f}(x+1) = \bar{f}(x) + 1$.

DEFINITION. Given $x \in R$, define the rotation number

$$\rho(\bar{f}, x) = \limsup_{n \to \infty} \frac{1}{n}(\bar{f}^n(x) - x).$$

Define the rotation set to be $\rho(\bar{f}) = \{\rho(\bar{f}, x) \mid x \in R\}$.

Notice that if a different lifting is used, this mearly has the effect of translating the rotation set by an integer.

We recall the following properties of $\rho(\bar{f})$.

(1) If $f = hgh^{-1}$ for an orientation preserving homeomorphism h of S, then $\rho(\bar{f}) = \rho(\bar{g})$ for suitable liftings $\bar{f}$ and $\bar{g}$.

(2) $\rho(\bar{f})$ is either one point or a closed interval.

(3) For any $\alpha \in \rho(\bar{f})$, there exists $x \in R$ such that $\lim_{n\to\infty}(\bar{f}^n(x) - x)/n = \alpha$.

(1) is trivial from the definition. See [2] and [3] for (2), (3) and other elementary properties of $\rho(\bar{f})$. By (2) we may denote the upper and lower endpoints of $\rho(\bar{f})$ by $\rho_+(\bar{f})$ and $\rho_-(\bar{f})$ respectively.

The aim of this paper is to prove the following

THEOREM 1. *Let M be the set of probability measures on S invariant with respect to f. Let $\varphi = \bar{f} - \mathrm{Id}$: $S \to R$ where* Id: $R \to R$ *denotes identity. Then $\rho(\bar{f}) = \{\mu(\varphi) \mid \mu \in M\}$.*

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Notes. (i) The set of probability measures on S is regarded as the set of positive linear functionals μ on $C(S)$ such that $\mu(1) = 1$.

(ii) S is considered to be $[0, 1]$ with 0 and 1 identified, and $\varphi(x) = \bar{f}(x) - x$ for $x \in [0, 1]$.

THEOREM 2. *If $\rho_+(\bar{f})$ $(\rho_-(\bar{f}))$ is irrational and $\theta > 0$, then $\rho_+(R_\theta \bar{f}) > \rho_+(\bar{f})$ $(\rho_-(R_\theta \bar{f}) > \rho_-(\bar{f}))$ where $R_\theta\colon R \to R$ is defined as $R_\theta(x) = x + \theta$.*

These two theorems are generalizations of properties given in Herman [**1**] for the case f is a homemorphism of S.

Considering (1) and Theorem 2, we obtain the following

COROLLARY. *If f is structurally stable, then both $\rho_+(\bar{f})$ and $\rho_-(\bar{f})$ are rational.*

PROOF OF THEOREM 1. By (3) above, for any $\alpha \in \rho(f)$, there exists $x \in S$ satisfying

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(\bar{f}^i(x)) = \lim_{n\to\infty} \frac{1}{n}(\bar{f}^n(x) - x) = \alpha$$

where $\varphi = \bar{f} - \mathrm{Id}$.

Define $\mu_n\colon C(S) \to R$ by $\mu_n(g) = n^{-1}\sum_{i=0}^{n-1} g(\bar{f}^i(x))$ for $g \in C(S)$ using x above. Then μ_n is a probability measure on S. Since the set of probability measures is weak* compact, there is a subsequence $\{\mu_{k_n}\}$ of $\{\mu_n\}$ which converges weakly to a probability measure, say μ. That is, we have $\mu_{k_n}(g) \to \mu(g)$ for any $g \in C(S)$. Taking $\varphi = \bar{f} - \mathrm{Id}$ for g, we have $\mu_{k_n}(\varphi) \to \mu(\varphi)$, while by definition,

$$\lim_{n\to\infty} \mu_{k_n}(\varphi) = \lim_{n\to\infty} \frac{1}{k_n} \sum_{i=0}^{k_n-1} \varphi(\bar{f}^i(x)) = \alpha.$$

Therefore, we have $\mu(\varphi) = \alpha$. Since $\lim_{n\to\infty}\mu_{k_n}(g) = \mu(g)$, μ is invariant with respect to f.

Consequently $\rho(\bar{f}) \subset \{\mu(\varphi) \mid \mu \in M\}$. On the other hand, for any rational $p/q < \rho_-(f)$, we have $\bar{f}^q(x) - x - p > 0$ for any $x \in S$. Thus $\mu(\bar{f}^q - \mathrm{Id} - q \cdot p/q) > 0$ for any $\mu \in M$, while

$$\mu(\bar{f}^q - \mathrm{Id} - q\mu(\varphi)) = \mu\left(\sum_{i=0}^{q-1} (\bar{f} - \mathrm{Id}) \circ f^i\right) - q\mu(\varphi) = 0.$$

Therefore $\mu(\varphi) > p/q$.

By the same reasoning, we have $\mu(\varphi) < p/q$ for any rational $p/q > \rho_+(\bar{f})$. Since $\rho(\bar{f})$ is a closed set and $\rho(\bar{f}) \subset \{\mu(\varphi) \mid \mu \in M\}$, we have $\rho(\bar{f}) = \{\mu(\varphi) \mid \mu \in M\}$. □

We need two lemmas to prove Theorem 2.

LEMMA 1. *For any irrational number α there exists a monotone decreasing sequence $\{p_n/q_n\}$ (and a monotone increasing sequence $\{p'_n/q'_n\}$) of rationals converging to α and satisfying $p_n/q_n - \alpha < 1/q_n^2$ $(\alpha - p'_n/q'_n < 1/q'^2_n)$.*

As this is a well-known fact in arithmetic, we do not give the proof.

LEMMA 2. *Let $\theta > 0$. For any $k \in N$ and any $x \in R$, there exists $y \in R$ such that $y \leqslant x$ and $(R_\theta \bar{f})^k(y) \geqslant \bar{f}^k(x) + \theta$.*

PROOF. We prove this by induction. For $k = 1$ it is trivial. Assume the lemma is true for k, then we have $y \leqslant x$ and $(R_\theta \bar{f})^k(y) \geqslant \bar{f}^k(x) + \theta$. Since $(R_\theta \bar{f})^k(s+n) = (R_\theta \bar{f})^k(s) + n$ for $n \in Z$, we have $z \leqslant y$ such that $(R_\theta \bar{f})^k(z) = \bar{f}^k(x)$, then

$$\begin{aligned}(R_\theta \bar{f})^{k+1}(z) &= \bar{f}\left((R_\theta \bar{f})^k(z)\right) + \theta \\ &= \bar{f}(\bar{f}^k(x)) + \theta = \bar{f}^{k+1}(x) + \theta\end{aligned}$$

and $z \leqslant x$, completing the induction. □

PROOF OF THEOREM 2. We prove the theorem for $\rho_+(\bar{f})$ because the $\rho_-(\bar{f})$ case is similar. By Lemma 1, we may choose a sequence $\{p_n/q_n\}$ of rationals such that $\{p_n/q_n\} \downarrow \alpha$ and $p_n/q_n - 1/q_n^2 < \alpha$. Since $\rho_+(\bar{f}) = \alpha$, we have $\bar{f}^{q_n}(x) - x < p_n$ for any $x \in R$. Suppose that there exists $\theta > 0$ such that $\bar{f}^{q_n}(x) - x < p_n - \theta$ for any $n \in N$ and any $x \in R$. Take q_n large enough to satisfy $q_n\theta > 1$, then we have

$$\bar{f}^{q_n^2}(x) - x = \sum_{i=0}^{q_n - 1} \left\{\bar{f}^{q_n}(\bar{f}^{iq_n}(x)) - \bar{f}^{iq_n}(x)\right\} < q_n(p_n - \theta) < q_n p_n - 1.$$

Thus, we have $\rho_+(\bar{f}) < (q_n p_n - 1)/q_n^2 < \alpha$, contradicting $\rho_+(\bar{f}) = \alpha$. Therefore, for any $\theta > 0$, there exist $n \in N$ and $x \in R$ such that $\bar{f}^{q_n}(x) - x \geqslant p_n - \theta$. On the other hand, by Lemma 2, there exists $y \leqslant x$ satisfying $(R_\theta \bar{f})^{q_n}(y) \geqslant \bar{f}^{q_n}(x) + \theta$. Thus, we have $(R_\theta \bar{f})^{q_n}(y) - y > \bar{f}^{q_n}(x) - x + \theta \geqslant p_n$. Therefore, $\rho_+(R_\theta \bar{f}) \geqslant p_n/q_n > \alpha$, completing the proof. □

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