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COBORDISM AND THE NONFINITE HOMOTOPY TYPE OF SOME DIFFEOMORPHISM GROUPS

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ABSTRACT. Unoriented cobordism, a geometric construction, and a theorem of Browder on finite H-spaces are used to give new examples of manifolds whose diffeomorphism groups have identity component of nonfinite homotopy type.

The nature of the group $\mathrm{Diff}(M)$ of smooth diffeomorphisms of a smooth manifold is of considerable current interest. It is known in many cases, and expected for most M, that $\mathrm{Diff}_0(M)$, the identity component of $\mathrm{Diff}(M)$ under the C^∞ topology, is not of finite homotopy type **[1]**. Our purpose is to give a simple construction of examples of this phenomenon.

The existence of these examples follows directly from a result in the theory of H-spaces, a geometrical construction, and a calculation in the unoriented cobordism ring.

FACT 1. Let X be an arcwise connected H-space. Then if X is of finite homotopy type, $\pi_2(X) = 0$ **[2]**.

Let $\mathfrak{N}_*$ denote the unoriented cobordism ring. An element $[M]$ of $\mathfrak{N}_*$ is said to fiber over the n-sphere S^n if and only if there is a representative M of $[M]$ and a smooth fiber bundle $p\colon M \to S^n$. Denote by $\mathfrak{N}^n$ the subset of elements of $\mathfrak{N}_*$ which fiber over S^n. Clearly $\mathfrak{N}^n$ is an ideal of $\mathfrak{N}_*$.

FACT 2. $\mathfrak{N}^{n+1} \subset \mathfrak{N}^n$.

The proof is by the following construction due to H. Winkelnkemper. Suppose $[M] \in \mathfrak{N}^{n+1}$ is represented by $F \xrightarrow{i} M \xrightarrow{p} S^{n+1}$. As $0 = [F \times S^{n+1}] \in \mathfrak{N}^{n+1}$ by considering $M + F \times S^{n+1}$ (disjoint union), M is represented by

$$F \times \{-1, 1\} \to M + F \times S^{n+1} \to S^{n+1},$$

where

$$F \times \{-1, 1\} = \partial(F \times [-1, 1]).$$

Now

$$M + F \times S^{n+1} = F \times \{-1, 1\} \times D^{n+1} \cup F \times \{-1, 1\} \times D^{n+1} \\ (F \times \{-1, 1\} \times S^n, \tilde{g}),$$

where $g\colon (S^n, *) \to (\mathrm{Diff}(F), \mathrm{id})$ is a smooth map and $\tilde{g}\colon F \times \{-1, 1\} \times S^n \to F \times \{-1, 1\} \times S^n$ is given by $\tilde{g}(x, -1, s) = (g(s)(x), -1, s)$ and $\tilde{g}(x, 1, s) = (x, 1, s)$.

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Extend $\tilde{g}$ to $\bar{g}$ on $F \times ([-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1]) \times S^n$ by $\bar{g}(x, -t^2, s) = (g(s)(x), -t^2, s)$ and $\bar{g}(x, t^2, s) = (x, t^2, s)$. The manifold

$$N = F \times [-1,1] \times D^{n+1} \underset{(F \times ([-1,-\frac{1}{2}] \cup [\frac{1}{2},1]) \times S^n, \bar{g})}{\cup} F \times [-1,1] \times D^{n+1}$$

whose boundary is the disjoint union of $M \cup F \times S^{n+1}$ and a fiber bundle over S^n with fiber

$$F \times [-\tfrac{1}{2}, \tfrac{1}{2}] \underset{(\partial(F \times [-\frac{1}{2}, \frac{1}{2}]), \mathrm{id})}{\cup} F \times [-\tfrac{1}{2}, \tfrac{1}{2}] = F \times S^1$$

is the required cobordism.

We now describe the relation between the clutching functions for the boundary of N. Denote by $\Omega_s(\mathrm{Diff}(F))$ the appropriately topologized space of smooth maps of $(S^1, *)$ to $(\mathrm{Diff}(F), \mathrm{id})$. Then there is an obvious map

$$e: \Omega_s(\mathrm{Diff}(F)) \to \mathrm{Diff}(F \times S^1)$$

given by $e(l)(x, t) = (l(t)(x), t)$. Denoting by $\Omega(\mathrm{Diff}(F))$ the loop space of $\mathrm{Diff}_0(F)$, i: $\Omega_s(\mathrm{Diff}(F)) \to \Omega(\mathrm{Diff}(F))$ is a homotopy equivalence. Regarding $\pi_i(\mathrm{Diff}_0(F))$ as $\pi_{i-1}\Omega_s(\mathrm{Diff}(F))$, we have the map

$$\tilde{e}: \pi_i(\mathrm{Diff}_0(F)) \to \pi_{i-1}(\Omega_s \mathrm{Diff}(F)) \xrightarrow{e_*} \pi_{i-1}\mathrm{Diff}(F \times S^1).$$

Given a smooth fiber bundle $F \to M \to S^{n+1}$, $n \geqslant 1$, determined by a clutching class $g \in \pi_n(\mathrm{Diff}_0(F))$, the bundle over S^n constructed above has fiber $F \times S^1$ and clutching class $\tilde{e}(g) \in \pi_{n-1}\mathrm{Diff}(F \times S^1)$.

OBSERVATION 3. Let $0 \neq [M] \in \mathfrak{R}^3$ and let $F \to M \to S^3$ be a representative of M with clutching class g. Then $0 \neq [g] \in \pi_2 \mathrm{Diff}_0(F)$.

Thus in order to show the existence of manifolds F with $\mathrm{Diff}_0(F)$ of nonfinite homotopy type, it suffices to show $0 \neq \mathfrak{R}^i$ for some $i \geqslant 3$.

As $\mathfrak{R}_*$ is a polynomial ring over Z_2, ϕ^2: $\mathfrak{R}_* \to \mathfrak{R}_*$; ϕ^2: $x \mapsto x^4$ is an injective homomorphism. Let $K_1 \subset \mathfrak{R}_*$ be the kernel of χ: $\mathfrak{R}_* \to Z_2$, where χ is the mod 2 Euler characteristic.

FACT 4. $\phi^2(K_1) \subset \mathfrak{R}^4$ [**3**, 7.3]. Hence $0 \neq \mathfrak{R}^4 \subset \mathfrak{R}^3$.

The generators of $\phi^2[K_1]$ represented by elements of $\mathfrak{R}^4$ which are given in §7 of [**3**] are all determined by $S^3 = \mathrm{Sp}(1)$ actions on manifolds with evidently nontrivial rational pontryagin classes so that these examples differ from those of [**1**].

It is worthwhile to consider the examples of [**3**] more explicitly. Let **F** be **R**, **C** or **H**, let $G(\mathbf{F})$ be the group of unit norm elements of F, and let $S(k\mathbf{F})$ denote the unit sphere in $k\mathbf{F}$. $G(\mathbf{F})^{k+1}$ denotes the $(k+1)$-fold direct product of $G(\mathbf{F})$, and $\Sigma^k(\mathbf{F})$ denotes the k-fold direct product of $S(2\mathbf{F})$. We define a $G(\mathbf{F})^{k+1}$ action on $\Sigma^k(\mathbf{F}) \times S((n+1)\mathbf{F})$ by

$$\begin{aligned}(t_1, \ldots, t_{k+1})&((q_1, p_1), \ldots, (q_k, p_k), (\rho_1, \ldots, \rho_{n+1}))\\ &= \left((q_1 t_1^{-1}, p_1 t_1^{-1}), \ldots, (q_j t_j^{-1}, t_{j-1} p_j t_j^{-1}), \ldots, (q_k t_k^{-1}, t_{k-1} p_k t_k^{-1})\right)\\ &\quad \left(t_k \rho_1 t_{k+1}^{-1}, \rho_2 t_{k+1}^{-1}, \ldots, \rho_{n+1} t_{k+1}^{-1}\right)\end{aligned}$$

Denote the quotient manifold of this principal action by $V(n, k; \mathbf{F})$. A point in the $p(n+k)$ manifold $V(n, k; \mathbf{F})$ $(p = 1, 2, 4)$ is denoted $[(q_1, p_1), \ldots, (q_k, p_k), (\rho_1, \ldots, \rho_{n+1})]$. The map $V(n, k; \mathbf{F}) \to \mathbf{F}P(1) = S^1, S^2, S^4$ given by

$$[(q_1, p_1), \ldots, (q_k, p_k); (\rho_1, \ldots, \rho_{n+1})] \to [(\rho_1, q_1)]$$

is a fiber map with fiber $V(n, k-1; \mathbf{F})$ and structure group $G(\mathbf{F})$ where the action of $G(\mathbf{F})$ on $V(n, k-1; \mathbf{F})$ is given by

$$\begin{aligned} t[(q_1, p_1), \ldots, (q_{k-1}, p_{k-1}), (\rho_1, \ldots, \rho_{n+1})] \\ = [(q_1, tp_1), (q_2, p_2), \ldots, (q_{k-1}, p_{k-1}), (\rho_1, \ldots, \rho_{n+1})]. \end{aligned}$$

Connor and Floyd established the following results, where $[M]$ denotes the class of M in $\mathfrak{N}_*$.

THEOREM. $[V(n, k; \mathbf{C})] = [V(n, k; \mathbf{R})]^2$, $[V(n, k; \mathbf{Q})] = [V(n, k; \mathbf{C})]^2$, $[V(2, 2p; \mathbf{R})]$ *is indecomposable in* $\mathfrak{N}_{2(p+1)}$.

As a straightforward result of Wall's computation of Ω_*, oriented bordism [4] we have

PROPOSITION. *Let* $F\colon \Omega_* \to \mathfrak{N}_*$ *be the forgetful map. If* $x \in \mathfrak{N}_*$ *is a power of an even indecomposable then* $x \notin F(\operatorname{tor} \Omega_*)$.

The manifold $V(n, k; \mathbf{H})$ is orientable; choosing an orientation, let $\{V(n, k; \mathbf{H})\}$ denote its class in Ω_*. From the results just quoted we have $\{V(2, 2p; \mathbf{H})\}$ is not torsion. Note that the Winkelnkemper construction produces oriented bordisms from oriented bundles. Also note that for oriented manifolds F^k the map

$$\pi_i \operatorname{Diff}_0(F) \to \pi_{i+1} \operatorname{B} \operatorname{Diff}_0(F) \to \Omega_{i+k+1}$$

that takes a fiber bundle over S^{i+1} to the oriented cobordism class of its total space is a homomorphism. Thus the elements of $\pi_2 \operatorname{Diff}_0(V(2, 2(p-1); \mathbf{H}) \times S^1)$ produced by the Wilkelnkemper construction on $V(2, 2p; \mathbf{H}) \to S^4$ are not torsion. We may then appeal to the classical theorem of Hopf, rather than that of Browder, in asserting the nonfinite homotopy type of $\operatorname{Diff}_0(V(2, 2(p-1); \mathbf{H}) \times S^1)$.

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