

GELFAND-KIRILLOV DIMENSION OF RINGS OF FORMAL DIFFERENTIAL OPERATORS ON AFFINE VARIETIES

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ABSTRACT. Let A be the coordinate ring of a smooth affine algebraic variety defined over a field k . Let D be the module of k -linear derivations on A and form $A[D]$, the ring of differential operators on A , as follows: consider A and D as subspaces of $\text{End}_k A$ (A acting by left multiplication on itself), and define $A[D]$ to be the subalgebra generated by A and D . Let $\text{rk } D$ denote the torsion-free rank of D (that is, $\text{rk } D = \dim_F F \otimes_A D$ where F is the quotient field of A). The ring $A[D]$ is a finitely generated k -algebra so its Gelfand-Kirillov dimension $\text{GK}(A[D])$ may be defined. The following is proved.

THEOREM. $\text{GK}(A[D]) = \text{tr deg}_k A + \text{rk } D = 2 \text{tr deg}_k A$.

Actually we work in a more general setting than that just described, and although a more general result is obtained, this is the most natural and important application of the main theorem.

1. Introduction. Let A be a finitely generated commutative algebra over the field k . In the terminology of [9] let D be a (k, A) -Lie algebra. We recall the definition.

- (i) D is a Lie algebra over k , with the Lie product denoted $[,]$;
- (ii) D is an A -module;
- (iii) there is an A -module homomorphism, $\theta: D \rightarrow \text{Der } A$ (the module of k -linear derivations on A); we denote $\theta(d)(a)$ by $d(a)$ for $d \in D, a \in A$;
- (iv) these structures are related by the requirement that $[d_1, ad_2] = a[d_1, d_2] + d_1(a)d_2$ for $d_1, d_2 \in D$ and $a \in A$.

The most natural examples of (k, A) -Lie algebras are simply submodules D of $\text{Der } A$, which are closed under the Lie bracket on $\text{Der } A$.

Given a (k, A) -Lie algebra D we form the ring of *formal differential operators*, $A\langle D \rangle$, as follows: it is the factor ring, $T_A(D)/J$, of the tensor algebra of the A -module D , by the ideal J generated by the relations $da - ad = d(a)$ for all $a \in A$, $d \in D$ and $d_1d_2 - d_2d_1 = [d_1, d_2]$ (for all $d_1, d_2 \in D$). When D is an abelian Lie algebra and a finitely generated free A -module the ring $A\langle D \rangle$ has been studied by a number of authors [1], [3–5].

We shall always assume D is a finitely generated A -module, in which case $A\langle D \rangle$, being a factor of $T_A(D)$, is a finitely generated k -algebra. Thus the Gelfand-Kirillov

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dimension of $A\langle D \rangle$ may be defined to be

$$\text{GK}(A\langle D \rangle) = \limsup_{n \rightarrow \infty} (\log \dim W^n / \log n)$$

where W is a finite-dimensional subspace of $A\langle D \rangle$ generating $A\langle D \rangle$ as a k -algebra. We adopt the convention that any subspace generating a k -algebra actually contains the field k ; W^n denotes the linear span of all words $w_1 \cdots w_n$ with each $w_i \in W$.

For any integral domain C with quotient field F , and any finitely generated C -module M , define the rank of M by $\text{rk } M = \dim_F F \otimes_C M$.

The problem is to determine $\text{GK}(A\langle D \rangle)$ in terms of other invariants of A and D . We prove

THEOREM A. *If D is a projective A -module and k is of characteristic 0, then*

$$\begin{aligned} \text{GK}(A\langle D \rangle) &= \max \left\{ \text{tr deg}_k(A/P) + \text{rk}_{A/P}(D/PD) \mid P \text{ is a minimal prime of } A \right\} \\ &= \text{tr deg}_k S_A(D), \end{aligned}$$

where $S_A(D)$ is the symmetric algebra of D .

The statement of the theorem simplifies when A is a domain or D is free: if A is a domain the assumption on the characteristic of k is unnecessary and $\text{GK}(A\langle D \rangle) = \text{tr deg } A + \text{rk } D$; if D is free then $\text{GK}(A\langle D \rangle) = \text{tr deg } A + \text{rk } D$. When D is free and the image of the generators of D in $\text{Der } A$ span a finite dimensional solvable Lie algebra it follows easily from [8] that $\text{GK}(A\langle D \rangle) = \text{tr deg } A + \text{rk } D$.

An earlier version of this paper benefitted from the referee's criticism and we would like to express our thanks.

2. Generalities concerning $A\langle D \rangle$. The natural map $A \oplus D \rightarrow A\langle D \rangle$ is an embedding and we identify A and D with their images in $A\langle D \rangle$. The ring $A\langle D \rangle$ is endowed with a natural filtration by the subspaces $R_n = (A + D)^n = A + D + D^2 + \cdots + D^n$. Using the Lie bracket on D it is easy to check that if $x \in R_n$ and $y \in R_m$ then $xy - yx \in R_{n+m-1}$. From this it follows that the associated graded algebra $\text{gr } A\langle D \rangle = \sum_{n=0}^{\infty} R_n / R_{n-1}$ is commutative. Furthermore, $\text{gr } A\langle D \rangle$ is generated as an algebra over A by the finitely generated A -module $R_1 / R_0 \cong D$, so there is a canonical surjection $S_A(D) \rightarrow \text{gr } A\langle D \rangle$, where $S_A(D)$ is the symmetric algebra of the A -module D .

PROPOSITION 2.1 [9, THEOREM 3.1]. *If D is projective, the canonical map $S_A(D) \rightarrow \text{gr } A\langle D \rangle$ is an isomorphism.*

PROOF. Just observe that $A\langle D \rangle$ coincides with the ring $V(A, D)$ of [9] (or the ring V_A of [7]).

Recall the definition [6] of the ring D_X of differential operators on an affine algebraic variety X .

PROPOSITION 2.2 [11]. *Let X be a smooth affine algebraic variety with coordinate ring A . Then $\text{Der } A$ is a projective A -module and D_X coincides with $A\langle \text{Der } A \rangle$. Furthermore, the filtration on D_X given by the order of the differential operators coincides with the filtration on $A\langle \text{Der } A \rangle$ given by the powers of the subspace $A + \text{Der } A$.*

PROOF. By [11, Theorem 18.2], as X is smooth, D_X is the subalgebra of $\text{End}_k A$ generated by A and $\text{Der}_k A$, so there is a natural surjection $\phi: A\langle \text{Der } A \rangle \rightarrow D_X$ and ϕ preserves the filtration. Hence, ϕ induces a map $\text{gr } \phi: \text{gr } A\langle \text{Der } A \rangle \rightarrow \text{gr } D_X$. But by Proposition 2.1, $\text{gr } A\langle \text{Der } A \rangle \cong S_A(\text{Der } A)$ and by [11, Theorem 18.2], $\text{gr } D_X \cong S_A(\text{Der } A)$ also. Hence $\text{gr } \phi$ is an isomorphism. The commutativity of the diagram

$$\begin{array}{ccc} A\langle \text{Der } A \rangle & \xrightarrow{\phi} & D_X \\ \downarrow & & \downarrow \\ \text{gr } A\langle \text{Der } A \rangle & \xrightarrow{\text{gr } \phi} & \text{gr } D_X \end{array}$$

(where the vertical maps are the gradings) ensures that ϕ is injective and hence an isomorphism.

COROLLARY 2.3. *If X is a smooth affine algebraic variety then $\text{GK}(D_X) = 2 \dim X$.*

PROOF. Just apply Theorem A, together with the standard fact that $\text{tr deg}_k A = \text{rk } \text{Der } A = \dim X$.

In [2, Chapter 2, §6] Bjork considers a noncommutative ring R equipped with a filtration such that the associated graded algebra is commutative and noetherian. An integer $d(R)$ is defined in terms of the properties of the associated graded algebra. The ring $A\langle D \rangle$ fits into this context and once Theorem A has been established it is easy to obtain the following

COROLLARY 2.4. *If D is projective, finitely generated and $\text{char } k = 0$, then $\text{GK}(A\langle D \rangle) = d(A\langle D \rangle)$.*

PROOF. The definition of d ensures that $d(A\langle D \rangle)$ equals the classical Krull dimension of $\text{gr } A\langle D \rangle$. But $\text{gr } A\langle D \rangle \cong S_A(D)$ by Proposition 2.1 and hence $d(A\langle D \rangle) = \text{tr deg } S_A(D) = \text{GK}(A\langle D \rangle)$ by Theorem A.

We do not know if $d(M)$ and $\text{GK}(M)$ coincide for an arbitrary finitely generated $A\langle D \rangle$ -module M . In the special case when $A\langle D \rangle$ is a Weyl algebra, Bjork [2, Chapter 3, §A.2.5] show that $d(M) = \text{GK}(M)$ for all finitely generated M .

3. Two lemmas concerning polynomials. The following is useful in the context of Hilbert polynomials.

LEMMA 3.1. *Let $f \in \mathbf{Q}[x]$. Then f is a polynomial of degree d if and only if the polynomial \tilde{f} , defined by $\tilde{f}(x) = f(x+1) - f(x)$, is a polynomial of degree $d-1$.*

We will need another lemma concerning polynomials.

LEMMA 3.2. *Let $p, q \in \mathbf{Q}[x]$ be polynomials of degree r, t , respectively, and suppose $c \in \mathbf{N}$ is fixed. Define a function f on \mathbf{N} by $f(n) = p(0)q(n) + p(1)q(n-1) + \dots + p(n-c)q(c)$. Then f is a polynomial of degree $r+t+1$.*

PROOF. By induction on the degree of q . If $\deg q = 0$ then q is a constant; say $q(n) = Q$. Thus, we have $f(n) = Q(p(0) + \dots + p(n-c))$ and $f(n+1) - f(n) = Qp(n+1-c)$. This is a polynomial of degree r , so by Lemma 3.1, f is a polynomial of degree $r+1$.

Suppose now that the result is true for polynomials of degree $t - 1$ and that $\deg q = t$. Then

$$f(n+1) - f(n) = \sum_{i=0}^{n-c} p(i)[q(n+1-i) - q(n-i)] + p(n+1-c)q(c).$$

The polynomial $q(n+1-i) - q(n-i)$ is of degree $t - 1$, so the induction hypothesis applied to the function $g(n) = \sum_{i=0}^{n-c} p(i)[q(n+1-i) - q(n-i)]$ ensures that g is of degree $r + t$. As $p(n+1-c)q(c)$ is of degree r , $f(n+1) - f(n)$ is a polynomial of degree $r + t$. Another application of Lemma 3.1 shows $f(n)$ is of degree $r + t + 1$.

4. Normal form of elements in $A\langle D \rangle$. Let C be a finite-dimensional generating subspace of D with basis d_1, \dots, d_r . Choose a finite-dimensional generating subspace V of A with the property that $[C, C] \subset VC$ and $C(V) \subset V^2$. It is possible to find such a subspace: just pick any subspace U of A , generating A and satisfying $[C, C] \subset UC$; for some l , $C(U) \subset U^l$, hence $C(U^l) \subset U^{2l}$, and putting $V = U^l$ we have a subspace with the required properties.

Notice that $C(V^n) \subset \sum_{j=0}^{n-1} V^j C(V) V^{n-j-1} \subset V^{n+1}$ for all n . Another way of expressing this is that $CV^n \subset V^n C + V^{n+1}$; we shall make frequent use of this fact.

Let C_n denote the k -linear span of $\{d_1^{i_1} \cdots d_r^{i_r} \mid i_1 + \cdots + i_r = n\}$ with the convention that $C_0 = k$; notice that $C = C_1$. It is clear that $R_n = AC_0 + AC_1 + \cdots + AC_n$; if an element of $A\langle D \rangle$ is written in the form

$$\sum a_{i_1 \dots i_r} d_1^{i_1} \cdots d_r^{i_r} \quad (\text{where each } a_{i_1 \dots i_r} \in A),$$

we shall say that the element is expressed in *normal form*.

Put $W = V \oplus C$; W is a finite-dimensional generating subspace of $A\langle D \rangle$. It is necessary to study $\dim W^n$ and as a first step towards this we will show that $W^n = \sum_{t=0}^n V^{n-t} C_t$ (Theorem 4.4). This result may be thought of as a statement about the normal form of elements in W^n , or as a statement about the product of two elements in normal form.

LEMMA 4.1. $C^p V \subset \sum_{j=0}^p V^{j+1} C^{p-j}$.

PROOF. It is true for $p = 1$ by what has already been said. Suppose that the statement is true for p . We shall prove it true for $p + 1$:

$$\begin{aligned} C^{p+1} V &\subset \sum_{j=0}^p CV^{j+1} C^{p-j} \subset \sum_{j=0}^p (V^{j+1} C + V^{j+2}) C^{p-j} \\ &\subset \sum_{j=0}^p V^{j+1} C^{p-j+1} + V^{j+2} C^{p-j} \subset \sum_{j=0}^{p+1} V^{j+1} C^{p-j+1}. \end{aligned}$$

LEMMA 4.2. $[C, C_m] \subset \sum_{j=0}^{m-1} V^{j+1} C^{m-j}$.

PROOF. Because $C_m \subset C^m$ we have

$$[C, C_m] \subset [C, C^m] \subset \sum_{i=0}^{m-1} C^i [C, C] C^{m-i-1}.$$

Hence $[C, C_m] \subset \sum_{i=0}^{m-1} C^i V C^{m-i}$ and the result follows at once from Lemma 4.1 applied to the term $C^i V$.

LEMMA 4.3. $C^p \subset \sum_{j=0}^{p-1} V^j C_{p-j}$.

PROOF. The result is true for $p = 1$. Suppose it is true for all integers less than or equal to p . Then

$$C^{p+1} \subset \sum_{j=0}^{p-1} CV^j C_{p-j} \subset \sum_{j=0}^{p-1} (V^j C + V^{j+1}) C_{p-j}.$$

We claim that for $m \leq p$, $CC_m \subset \sum_{i=0}^m V^i C_{m+1-i}$. This is true for $m = 0$, and we prove it by induction on m . Suppose it is true for $m - 1$. Let d_{j+1} be one of the basis elements of C , and pick $d_r^{i_1} \cdots d_r^{i_r} \in C_m$. It is enough to show that

$$x = d_{j+1}(d_1^{i_1} \cdots d_r^{i_r}) \in \sum_{i=0}^m V^i C_{m+1-i}.$$

Now

$$d_{j+1}(d_1^{i_1} \cdots d_r^{i_r}) = ([d_{j+1}, d_1^{i_1} \cdots d_j^{i_j}] - d_1^{i_1} \cdots d_j^{i_j} d_{j+1}) d_{j+1}^{i_{j+1}} \cdots d_r^{i_r}$$

is an element of $[C, C_t]C_{m-t} + C_{m+1}$ for some t , $m \geq t \geq 1$. By Lemma 4.2 we see that

$$\begin{aligned} x &\in \sum_{i=0}^{t-1} V^{i+1} C^{t-i} C_{m-t} + C_{m+1} \subset \sum_{i=0}^{t-1} V^{i+1} C^{m-i} + C_{m+1} \\ &\subset \sum_{i=0}^{m-1} V^{i+1} C^{m-i} + C_{m+1}. \end{aligned}$$

If we now apply the induction hypothesis of the lemma to C^{m-i} for each i ($0 \leq i \leq m - 1$), we have that

$$\begin{aligned} x &\in \sum_{i=0}^{m-1} V^{i+1} \left(\sum_{j=0}^{m-i-1} V^j C_{m-i-j} \right) + C_{m+1} \\ &= \sum_{i=0}^{m-1} \sum_{j=0}^{m-i-1} V^{i+j+1} C_{m-i-j} + C_{m+1}. \end{aligned}$$

It is now clear that the claim is true for m .

Returning to the lemma, and applying the claim we have established that

$$C^{p+1} \subset \left(\sum_{j=0}^{p-1} V^j \sum_{i=0}^{p-j} V^i C_{p-j+1-i} \right) + \sum_{j=0}^{p-1} V^{j+1} C_{p-j}.$$

The truth of the lemma for $p + 1$ follows.

THEOREM 4.4. $W^n = \sum_{t=0}^n V^{n-t} C_t$.

PROOF. We proceed by induction, the case $n = 1$ being true from the definition of W . It is clear that $W^n \supset \sum_{t=0}^n V^{n-t} C_t$ as $C_t \subset W^t$, and $V^{n-t} \subset W^{n-t}$. We prove the

reverse inclusion. Suppose it is true for n . Then $W^{n+1} = \sum_{t=0}^n V^{n+1-t}C_t + \sum_{t=0}^n CV^{n-t}C_t$. The first of these terms belongs to $\sum_{t=0}^{n+1} V^{n+1-t}C_t$, and it remains only to prove that the second also does. By Lemma 4.3, and the fact that $CC_t \subset C^{t+1}$ we have

$$\begin{aligned} \sum_{t=0}^n CV^{n-t}C_t &\subset \sum_{t=0}^n (V^{n-t}C + V^{n-t+1})C_t \\ &\subset \sum_{t=0}^n V^{n-t} \left(\sum_{j=0}^t V^j C_{t+1-j} \right) + \sum_{t=0}^{n+1} V^{n+1-t}C_t \\ &\subset \sum_{t=0}^{n+1} V^{n+1-t}C_t. \end{aligned}$$

5. Proof of the theorem.

LEMMA 5.1. *If $\text{gr } A\langle D \rangle \cong S_A(D)$ then $\text{GK}(A\langle D \rangle) \geq \text{GK}(S_A(D))$.*

PROOF. Pick V, C, W as in §4, so that $W^n = \sum_{t=0}^n V^{n-t}C_t$. Now $S_A(D)$ is generated as a k -algebra by the subspace $V + \bar{C}$, where $\bar{C} = C + R_0/R_0$. So to evaluate $\text{GK}(S_A(D))$ we must examine $\dim(V + \bar{C})^n$. But

$$\begin{aligned} \dim(V + \bar{C})^n &= \dim \sum_{t=0}^n V^{n-t}(\bar{C})^t = \dim \sum_{t=0}^n V^{n-t}(C_t + R_{t-1}/R_{t-1}) \\ &= \dim \sum_{t=0}^n (V^{n-t}C_t + R_{t-1}/R_{t-1}) \\ &\leq \dim \sum_{t=0}^n V^{n-t}C_t = \dim W^n \end{aligned}$$

(by Theorem 4.4), and the lemma follows from the inequality $\dim(V + \bar{C})^n \leq \dim W^n$.

We shall first obtain $\text{GK}(A\langle D \rangle)$ under the assumption that A is a domain. The general case will be reduced to the domain case by considering A/P for the minimal primes P of A .

LEMMA 5.2. *If A is a domain and D a torsion-free A -module then*

$$\text{GK}(S_A(D)) = \text{tr deg}_k A + \text{rk } D.$$

PROOF. As remarked in [10], under the hypotheses of the lemma, $S_A(D)$ is a torsion-free A -module; in particular, regular elements of A remain regular as elements of $S_A(D)$. Hence, the natural map $S_A(D) \rightarrow F \otimes_A S_A(D)$ is an embedding and the latter may be considered as lying in the quotient field of $S_A(D)$. In a commutative ring the GK-dimension coincides with the transcendence degree, so $\text{GK}(S_A(D)) = \text{tr deg}_k(F \otimes_A S_A(D))$. It is standard that $F \otimes_A S_A(D) \cong S_F(F \otimes_A D)$; but $F \otimes_A D$ is just a free F -module on $r = \text{rk } D$ generators, so $S_F(F \otimes_A D) \cong F[X_1, \dots, X_r]$, the polynomial extension in r indeterminates. The transcendence degree of $F[X_1, \dots, X_r]$ is simply $\text{tr deg}_k F + r$; whence the result.

Combining these two lemmas gives half of Theorem A (at least for A a domain) as any projective module is certainly torsion free.

PROPOSITION 5.3. *If A is a domain and D torsion free, then*

$$\mathrm{GK}(A\langle D \rangle) \leq \mathrm{tr deg} A + \mathrm{rk} D.$$

PROOF. Pick $0 \neq x \in A$ such that, if $B = A[x^{-1}]$ then $E = B \otimes_A D$ is a free B -module of rank $r = \mathrm{rk} D$. As D is torsion free, so too is $S_A(D)$, and hence $A\langle D \rangle$ itself is a torsion-free A -module. In particular, x is a regular element of $A\langle D \rangle$ so the natural map $A\langle D \rangle \rightarrow B\langle E \rangle$ is injective, and it is enough to prove the proposition for $B\langle E \rangle$. So assume D is a free A -module.

Pick W, V, C as in §4, but with the extra condition that C is a vector space of dimension $r = \mathrm{rk} D$. As $\mathrm{gr} A\langle D \rangle \cong S_A(D) \cong A[X_1, \dots, X_r]$, one has for all t that $\dim C_t = \binom{t+r}{r}$. Because $W^n = \sum_{t=0}^n V^{n-t}C_t$,

$$\dim W^n \leq \sum_{t=0}^n (\dim V^{n-t})(\dim C_t) = \sum_{t=0}^n q(n-t)p(t),$$

say, where $q(n-t) = \dim V^{n-t}$, $p(t) = \dim C_t$. However, $p(t)$ is a polynomial of degree r , and $q(n-t)$ is a polynomial of degree $d = \mathrm{tr deg} A$. Hence by Lemma 3.2, $\sum_{t=0}^n q(n-t)p(t)$ is a polynomial of degree $d+r$. Because $\dim W^n$ is bounded above by a polynomial of degree $\mathrm{tr deg} A = \mathrm{rk} D$, the result follows.

Now we have the upper bound for $\mathrm{GK}(A\langle D \rangle)$, and combining the previous three lemmas proves Theorem A for A a domain. Notice that to prove the theorem for a domain no assumption on $\mathrm{char} k$ was required. The necessity of the condition becomes clear in the following (where we no longer assume A is a domain).

PROPOSITION 5.4. *Let A be an algebra over a field of characteristic zero. Let D be a projective (k, A) -Lie algebra. Then*

$$\mathrm{GK}(A\langle D \rangle) = \max \left\{ \mathrm{GK} \left(\frac{A}{P} \langle D/DP \rangle \right) \middle| P \text{ is a minimal prime of } A \right\}.$$

PROOF. When A is an algebra over a field of characteristic zero then the minimal primes of A are invariant under every derivation on A . There are minimal primes P_1, \dots, P_n of A with $P_1 \cdots P_n = 0$. Putting $R = A\langle D \rangle$ and using the fact that $RP_i = P_iR$ is an ideal of R , one has the product $(P_1R)(P_2R) \cdots (P_nR)$ equal to zero. Consequently, $\mathrm{GK}(A\langle D \rangle) = \max\{\mathrm{GK}(R/P_iR) \mid i = 1, \dots, n\}$. However, given an ideal I of A , invariant under any derivation, $R/IR \cong (A/I)\langle D/DI \rangle$. This follows from the fact that the diagram

$$\begin{array}{ccc} A\langle D \rangle & \rightarrow & (A/I)\langle D/DI \rangle \\ \downarrow & & \downarrow \\ S_A(D) & \rightarrow & S_{A/I}(D/DI) \end{array}$$

(with the vertical maps being the gradings and the horizontal maps being those induced by the natural surjections $A \rightarrow A/I$, $D \rightarrow D/DI$) is commutative, and the kernel of the lower map is the ideal of $S_A(D)$ generated by I .

COROLLARY 5.5. *Let A and D be as above. Then*

$$\mathrm{GK}(A\langle D \rangle) = \max \{ \mathrm{tr deg}_k A/P + \mathrm{rk}_{A/P}(D/DP) \mid P \text{ is a minimal prime of } A \}.$$

PROOF. Just use the above propositions, and note that if D is projective as an A -module then D/DP is projective as an A/P -module. This completes the proof of Theorem A.

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