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LIE IDEALS AND JORDAN DERIVATIONS OF PRIME RINGS

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ABSTRACT. Herstein proved [1, Theorem 3.3] that any Jordan derivation of a prime ring of characteristic not 2 is a derivation of R. Our purpose is to extend this result on Lie ideals. We prove the following

THEOREM. *Let R be any prime ring such that* char $R \neq 2$ *and let U be a Lie ideal of R such that $u^2 \in U$ for all $u \in U$. If, ′, is an additive mapping of R into itself satisfying $(u^2)' = u'u + uu'$ for all $u \in U$, then $(uv)' = u'v + uv'$ for all $u, v \in U$.*

Introduction. Herstein [**1**, Theorem 3.3] proved that if R is a prime ring of characteristic different from 2, then any Jordan derivation of R, i.e., an additive mapping of R into itself such that $(a^2)' = a'a + aa'$ for all $a \in R$, is a derivation of R, i.e., an additive mapping of R into itself such that $(ab)' = a'b + ab'$ for all a, $b \in R$. In this paper we generalize this result on Lie ideals.

Throughout the paper we assume R is a prime ring of characteristic not 2. The center of R is denoted by Z. We always assume U is a Lie ideal of R with the condition that $u^2 \in U$ for all $u \in U$. We also assume, ′, is an additive mapping of R into itself such that

(i) $(u^2)' = u'u + uu'$ for all $u \in U$.

Note that $(uv + vu) = (u + v)^2 - (u^2 + v^2)$. Hence $(uv + vu)$ is in U and condition (i) is equivalent to

(ii) $(uv + vu)' = u'v + uv' + v'u + vu'$ for all $u, v \in U$.

For $x, y \in R$, let

$$[x, y] = xy - yx \quad \text{and} \quad x^y = (xy)' - x'y - xy'.$$

If A is a subset of R, we define the centralizer of A

$$C_R(A) = \{x \in R \mid [x, a] = 0 \text{ for all } a \in A\}.$$

An additive subgroup U of R is said to be a Lie ideal of R if $[u, r] \in U$ for all $u \in U$ and $r \in R$. For the remainder of the paper, the letters u, v, w, u_i, v_i, w_i will always denote arbitrary elements in U.

If U is a commutative Lie ideal of R, then by the proof of Lemma 1.3 [**1**], $U \subset Z$. Then from (ii) we get $2(uv)' = 2(u'v + uv')$. Since char $R \neq 2$, we get the desired conclusion.

Thus we shall always assume U is a noncommutative Lie ideal of R, i.e., $U \not\subset Z$.

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2. Basic lemmas.

LEMMA 1. *If $U \not\subset Z$ is a Lie ideal of R, then*

$$(uvu)' = u'vu + uv'u + uvu' \quad \text{for all } u, v \in U.$$

PROOF. The proof is the same as that of Lemma 3.5 of [**1**], since $uv + vu \in U$ for $u, v \in U$.

By linearizing Lemma 1 on u, we get

LEMMA 2. *If $U \not\subset Z$ is a Lie ideal of R, then*

$$(uvw + wvu)' = u'vw + uv'w + uvw' + w'vu + wv'u + wvu' \quad \text{for all } u, v, w \in W.$$

LEMMA 3. *If $u \not\subset Z$ is a Lie ideal of R, then $u^v[u, v] = 0$ for all $u, v \in U$.*

PROOF. Since for any $u, v \in U$, $uv + vu \in U$ and also $uv - vu \in U$, as U is a Lie ideal, we have $2uv \in U$. Therefore, from (i), since char $R \neq 2$, we get

$$((uv)^2)' = (uv)'(uv) + (uv)(uv)'.$$

In Lemma 2 replace w by $2uv$ to get

$$\begin{aligned} 2(uv(uv) + (uv)vu)' &= 2(u'v(uv) + uv'(uv) + uv(uv)' \\ &\qquad + (uv)'vu + (uv)v'u + (uv)vu') \\ &= 2((u'v + uv')uv + (uv)'vu) + 2uv((uv)' + v'u + vu'). \end{aligned}$$

But

$$\begin{aligned} 2(uv(uv) + (uv)vu)' &= 2((uv)^2 + uv^2u)' \\ &= 2((uv)'uv + uv(uv)' + u'v^2u + u(v'v + vv')u + uv^2u') \\ &= 2((uv)'uv + (u'v + uv')vu) + 2uv((uv)' + v'u + vu'), \end{aligned}$$

by Lemma 1 and (i). After comparing both expressions, since char $R \neq 2$, we get

$$\{(uv)' - u'v - uv'\}(uv - vu) = 0, \quad \text{i.e.,} \quad u^v[u, v] = 0 \text{ for all } u, v \in U.$$

LEMMA 4. *If $U \not\subset Z$ is a Lie ideal of R, then $[u, v]u^v = 0$ for all $u, v \in U$.*

PROOF. Replace w by $2vu$ in Lemma 2 and continue by the same procedure as in Lemma 3 to get $[u, v]v^u = 0$. But in view of condition (ii), $u^v + v^u = 0$, i.e., $v^u = -u^v$. So we get the desired conclusion that $[u, v]u^v = 0$ for all $u, v \in U$.

LEMMA 5. *If $U \not\subset Z$ is a Lie ideal of R and, for $u \in U$, if $u \in C_R(U)$, then $u' \in Z$.*

PROOF. By [**2**, Lemma 2], $C_R(U) = Z$, so $u \in Z$. From (ii), we have

$$(2uv)' = (u'v + vu') + 2uv' \quad \text{for all } v \in U.$$

Replacing v by $vw + wv$ in the last equation, we get

$$(2u(vw + wv))' = (u'(vw + wv) + (vw + wv)u') + 2u(vw + wv)'.$$

Since $u \in Z$, by Lemma 2 we get

$$\begin{aligned} (2u(vw + wv))' &= 2(uvw + wvu)' \\ &= 2(u'vw + uv'w + uvw' + w'vu + wv'u + wvu') \\ &= 2(u'vw + wvu') + 2u(v'w + vw' + w'v + wv'). \end{aligned}$$

Compare the two expressions for $(2u(vw + wv))'$ to obtain

$$u'(vw - wv) = (vw - wv)u' \quad \text{for all } v, w \in U,$$

i.e., $u' \in C_R([U, U]) = C_R(U)$ by [**2**, Lemma 3]. But, as above, $C_R(U) = Z$, so $u' \in Z$.

LEMMA 6. *If $U \not\subset Z$ is a Lie ideal of R and, for $u, v \in U$, if $uv = vu$, then $u^v = 0$.*

PROOF. From Lemma 2, for all $w \in U$,

$$(uvw + wvu)' = u'vw + uv'w + uvw' + w'vu + wv'u + wvu'.$$

But by hypothesis $uv = vu$, so by (ii),

$$(uvw + wvu)' = (uv \cdot w + w \cdot uv)' = (uv)'w + (uv)w' + w'(uv) + w(uv)',$$

since $2uv \in U$ and char $R \neq 2$.

On comparing both expressions for $(uvw + wvu)'$, since $uv = vu$, we get

$$\{(uv)' - u'v - uv'\}w + w\{(vu)' - v'u - vu'\} = 0,$$

so $(u^v)w + w(v^u) = 0$. By (ii) $v^u = -u^v$, so $(u^v)w - w(u^v) = 0$ for all $w \in U$. Then $u^v \in C_R(U) = Z$ by [**2**, Lemma 2]. So we conclude that for u, $v \in U$, if $uv = vu$ then $u^v \in Z$, i.e., $(uv)' - u'v - uv' \in Z$. Since $u^2 \in U$ and $u^2v = vu^2$, then

$$(u^2v)' - (u^2)'v - u^2v' \in Z,$$

so

$$(u^2v)' - (u'u + uu')v - u^2v' \in Z$$

by (i). Again, as $2uv \in U$ and $u(2uv) = (2uv)u$, we get

$$(u(2uv))' - u'(2uv) - u(2uv)' \in Z,$$

i.e.,

$$(u^2v)' - u'(uv) - u(uv)' \in Z,$$

since char $R \neq 2$. Thus

$$u(u^v) = u\{(uv)' - u'v - uv'\} = \{(u^2v)' - (u'u + uu')v - u^2v'\} - \{(u^2v)' - u'(uv) - u(uv)'\} \in Z.$$

If $u^v \neq 0$, since R is prime and $u^v \in Z$, then we get $u \in Z$, so by Lemma 5, $u' \in Z$. Then by (ii), $u^v = 0$, a contradiction. Hence $u^v = 0$.

3. The main theorem. Now we are in position to prove the following theorem which extends a result of Herstein [**1**, Theorem 3.3].

THEOREM. *Let R be a prime ring,* char $R \neq 2$, *and let U be a Lie ideal of R such that $u^2 \in U$ for all $u \in U$. If $'$ is an additive mapping of R into itself such that $(u^2)' = u'u + uu'$ for all $u \in U$, then $(uv)' = u'v + uv'$ for all $u, v \in U$.*

PROOF. Linearizing Lemmas 3 and 4 on v, we get

(1) $$u^v[u, w] + u^w[u, v] = 0, \quad \text{i.e.,} \quad u^v[u, w] = -u^w[u, v]$$

and

(2) $$[u,w]u^v+[u,v]u^w=0, \quad \text{i.e.,} \quad [u,w]u^v=-[u,v]u^w.$$

Multiplying by $[u, w_1]$ on the left-hand side of (1) and using (2) and (1) we get

(3) $$[u,v]u^w[u,w_1]=-[u,w_1]u^w[u,v].$$

Let $w_1=2w_1v_1$ in (3); since char $R\neq 2$, we get

$$\begin{aligned}&[u,v]u^w[u,w_1]v_1+[u,v]u^w w_1[u,v_1]\\&\quad=-[u,w_1]v_1u^w[u,v]-w_1[u,v_1]u^w[u,v],\end{aligned}$$

or

(4) $$\begin{aligned}&[u,v]u^w[u,w_1]v_1+[u,w_1]v_1u^w[u,v]\\&\quad=-[u,v]u^w w_1[u,v_1]-w_1[u,v_1]u^w[u,v].\end{aligned}$$

Applying (1) and (2) to (3) we have

$$[u,v]u^w[u,w_1]=[u,w_1]u^v[u,w],$$
$$[u,w]u^v[u,w_1]=[u,w_1]u^w[u,v],$$

and using these in (4) we obtain

$$\begin{aligned}&[u,w_1]u^v[u,w]v_1+[u,w_1]v_1u^w[u,v]\\&\quad=-[u,v]u^w w_1[u,v_1]-w_1[u,w]u^v[u,v_1],\end{aligned}$$

or

$$[u,w_1]\{u^v[u,w]v_1+v_1u^w[u,v]\}=-\{[u,v]u^w w_1+w_1[u,w]u^v\}[u,v_1].$$

In view of (1) and (2), the last equation gives

$$\begin{aligned}&[u,w_1]\{u^v[u,w]\cdot v_1-v_1\cdot u^v[u,w]\}\\&\quad=-\{[u,v]u^w\cdot w_1-w_1\cdot[u,v]u^w\}[u,v_1],\end{aligned}$$

or

(5) $$[u,w_1][u^v[u,w],v_1]=-[[u,v]u^w,w_1][u,v_1].$$

In (5), replace v_1 by $2v_1u_1$ and use (5). Since char $R\neq 2$, we get

(6) $$[u,w_1]v_1[u^v[u,w],u_1]=-[[u,v]u^w,w_1]v_1[u,u_1].$$

Replace v_1 by $[u, w_1]$ in (6). Then

$$[u,w_1][u,w_1][u^v[u,w],u_1]=-[[u,v]u^w,w_1][u,w_1][u,u_1].$$

Write $v_1=u_1$ in (5). Then

$$[u,w_1][u^v[u,w],u_1]=-[[u,v]u^w,w_1][u,u_1],$$

and using this in the last equation we get

$$-[u,w_1][[u,v]u^w,w_1][u,u_1]=-[[u,v]u^w,w_1][u,w_1][u,u_1],$$

or

$$\{[[u,v]u^w,w_1][u,w_1]-[u,w_1][[u,v]u^w,w_1]\}[u,u_1]=0.$$

Replace u_1 by $2u_2u_1$ in the last equation and use it to get

$$\{[[u,v]u^w,w_1][u,w_1]-[u,w_1][[u,v]u^w,w_1]\}U[u,u_1]=0.$$

If, for some $u, u_1 \in U$, $u^{u_1} \neq 0$, then by Lemma 6 $[u, u_1] \neq 0$, so by [**2**, Lemma 4], we get

$$[[u, v]u^w, w_1][u, w_1] - [u, w_1][[u, v]u^w, w_1] = 0.$$

Write $v_1 = w_1$ in (5). Then in view of the last equation, we have

$$[u, w_1][u^v[u, w], w_1] = -[[u, v]u^w, w_1][u, w_1] = -[u, w_1][[u, v]u^w, w_1],$$

or

$$(7) \qquad [u, w_1][u^v[u, w] + [u, v]u^w, w_1] = 0.$$

Linearizing (7) on w_1 we have

$$(8) \quad [u, w_1][u^v[u, w] + [u, v]u^w, v_2] + [u, v_2][u^v[u, w] + [u, v]u^w, w_1] = 0.$$

Now replace w_1 by $2uw_1$ in (8). Since char $R \neq 2$ we get

$$u[u, w_1][u^v[u, w] + [u, v]u^w, v_2] + [u, v_2][u^v[u, w] + [u, v]u^w, u]w_1 + [u, v_2]u[u^v[u, w] + [u, v]u^w, w_1] = 0.$$

When $w_1 = u$ in (8) then $[u, v_2][u^v[u, w] + [u, v]u^w, u] = 0$. Thus from the last equation we get

$$u[u, w_1][u^v[u, w] + [u, v]u^w, v_2] + [u, v_2]u[u^v[u, w] + [u, v]u^w, w_1] = 0.$$

But again in view of (8), the last equation reduces to

$$-u[u, v_2][u^v[u, w] + [u, v]u^w, w_1] + [u, v_2]u[u^v[u, w] + [u, v]u^w, w_1] = 0,$$

or

$$(9) \qquad [u, [u, v_2]][u^v[u, w] + [u, v]u^w, w_1] = 0.$$

Replace w_1 by $2w_2w_1$ in (9) and use (9) to obtain

$$[u, [u, v_2]]U[u^v[u, w] + [u, v]u^w, w_1] = 0.$$

Then by [**2**, Lemma 4] either $[u, [u, v_2]] = 0$ or $[u^v[u, w] + [u, v]u^w, w_1] = 0$. If $[u, [u, v_2]] = 0$ for all $v_2 \in U$, then by the Corollary of Theorem 1 [**2**] $[u, U] = 0$, i.e., $u \in C_R(U) = Z$ by [**2**, Lemma 2], so by Lemma 6, $u^{u_1} = 0$, a contradiction. Hence

$$[u^v[u, w] + [u, v]u^w, w_1] = 0 \quad \text{for all } w_1 \in U,$$

i.e., $u^v[u, w] + [u, v]u^w \in C_R(U) = Z$. Thus, in view of (2), we get

$$(10) \qquad u^v[u, w] - [u, w]u^v \in Z.$$

Commuting (10) with u^v, since by (2) and Lemma 3, $u^v[u, w]u^v = 0$, then

$$(11) \qquad u^v u^v[u, w] + [u, w]u^v u^v = 0.$$

Commuting (10) with $[u, w]$, since by (1) and Lemma 4 $[u, w]u^v[u, w] = 0$, we get

$$(12) \qquad u^v[u, w]^2 + [u, w]^2 u^v = 0.$$

Let us set $\alpha = u^v[u, w]$ and $\beta = [u, w]u^v$. By (2) and Lemma 3, we get $u^v[u, w]u^v = -u^v[u, v]u^w = 0$. Thus $\alpha^2 = 0$. Similarly, we can show $\beta^2 = 0$. In view of (12) and (11) we have

$$\alpha\beta = u^v[u, w]^2 u^v = -[u, w]^2 u^v u^v = [u, w]u^v u^v[u, w] = \beta\alpha.$$

Now

$$(\alpha-\beta)^3=\alpha^3+\alpha\beta^2+\beta\alpha\beta+\beta^2\alpha-\alpha^2\beta-\alpha\beta\alpha-\beta\alpha^2-\beta^3=0,$$

since $\alpha^2=\beta^2=0$ and $\alpha\beta=\beta\alpha$. Since R is prime and by (10) $(\alpha-\beta)\in Z$, then $\alpha-\beta=0$, i.e., $\alpha=\beta$. Thus we get

$$u^v[u,w]=[u,w]u^v, \quad \text{i.e.,} \quad [u^v,[u,w]]=0. \tag{13}$$

Let $w=2wu_3$ in (13). Then

$$\begin{aligned}0&=[u^v,[u,2wu_3]]=2[u^v,[u,w]u_3+w[u,u_3]]\\&=2[u^v,[u,w]u_3]+2[u^v,w[u,u_3]]\\&=2[u^v,[u,w]]u_3+2[u,w][u^v,u_3]+2[u^v,w][u,u_3]+2w[u^v,[u,u_3]].\end{aligned}$$

By (13) the first and fourth terms are zero. Since char $R\neq 2$, from above we get $[u,w][u^v,u_3]+[u^v,w][u,u_3]=0$. Now replace w by $[u,w]$ and, in view of (13), we get $[u,[u,w]][u^v,u_3]=0$. Replace u_3 by $2v_3u_3$ to get

$$[u,[u,w]]U[u^v,u_3]=0.$$

By [**2**, Lemma 4], either $[u,[u,w]]=0$ or $[u^v,u_3]=0$. As above, we have seen that $[u,[u,w]]\neq 0$, therefore $[u^v,u_3]=0$ and so $u^v\in C_R(U)=Z$. By Lemma 3, $u^{u_1}[u,u_1]=0$. Since u^{u_1} $(\neq 0)\in Z$ and R is prime, we get $[u,u_1]=0$. Therefore by Lemma 6, $u^{u_1}=0$. Hence for all $u,v\in U$, $u^v=0$, i.e., $(uv)'=u'v+uv'$.

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