ON SYLOW INTERSECTIONS IN FINITE GROUPS

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Abstract. In general, for a given prime $p$ and finite group $G$, there need not be Sylow $p$-subgroups $P$ and $Q$ of $G$ with $P \cap Q = O_p(G)$. In this paper we show that if $G$ is $p$-soluble, and $p$ is not 2 or a Mersenne prime, then such Sylow $p$-subgroups exist (also we give conditions guaranteeing the existence of such Sylow subgroups when $p$ is 2 or a Mersenne prime). We also show that if $G$ is not $p$-soluble, but $p$ is odd and the components of $G/O_p(G)$ are in a certain class of quasi-simple groups, then there are Sylow $p$-subgroups $P$ and $Q$ of $G$ with $P \cap Q = O_p(G)$, unless perhaps $p$ is a Mersenne prime. When $G$ is $p$-soluble, our work extends results of N. Itô [2].

In general, when $G$ is a finite group and $p$ is a prime divisor of $|G|$, there need not be Sylow $p$-subgroups $P$ and $Q$ of $G$ such that $P \cap Q = O_p(G)$. The aim of this note is to prove that in many situations, though, there are such Sylow $p$-subgroups.

Our first observation is that “regular-orbit” theorems such as that of Hargreaves [1] can be used to establish the existence of such Sylow subgroups in $p$-solvable groups when $p \neq 2$ or a Mersenne prime. More precisely, we can prove the following theorem, which slightly strengthens a result of Itô [2].

Theorem 1. Let $G$ be a finite $p$-solvable group, and $P \in \text{Syl}_p(G)$. Then there is a Sylow $p$-subgroup, $Q$, of $G$ with $P \cap Q = O_p(G)$, unless, perhaps, one of the following situations occur:

(a) $p = 2$, $P/O_p(G)$ involves $Z_2 \sim Z_2$ and $|O_{p,p}(G)|$ is divisible by the square of a Fermat prime, or the square of a Mersenne prime.

(b) $p$ is a Mersenne prime, $P/O_p(G)$ involves $Z_p \sim Z_p$, $P/O_p(G)$ does not centralize $O_2(G/O_p(G))$, and $(p + 1)^p$ divides $|G|$.

Proof. We proceed by induction on $|G|$. Suppose that neither (a) nor (b) hold. Certainly we may suppose that $O_p(G) = 1_G$. By the Hall-Higman centralizer lemma, $C_G(O_p(G)) \leq O_p(G)$. Thus $O_p(PO_p(G)) = 1_G$, so we may suppose that $G = PO_p(G)$.

We claim that $O_p(G)$ is a $q$-group for some prime $q \neq p$. Let $(r_i; 1 \leq i \leq n)$ be the set of prime divisors of $|O_p(G)|$, and for each $i$, let $R_i$ be a $P$-invariant Sylow $r_i$-subgroup of $O_p(G)$. Suppose that $n > 1$. If $p$ is not a Mersenne prime, then neither (a) nor (b) hold in $PR_i$, so by induction there are some $x_i$ in $R_i$ with $P \cap P^{x_i} = C_p(R_i)$. Let $x = x_1x_2 \cdots x_n$. Then $x$ centralizes $P \cap P^x$. Let $u \in P \cap P^x$. Then $x^u = x$, so that $x_1^ux_2^u \cdots x_n^u = x_1x_2 \cdots x_n$. Now $x_i^u \in R_i$ for each $i$, and
each element of $O_p(G)$ has a unique expression of the form $z_1z_2 \cdots z_n$, where each $z_i \in R_i$. Thus $x_i^u = x_i$ for each $i$, so that $u \in P \cap P^x_i$ for each $i$. Consequently, $u \in \bigcap_{i=1}^n C_p(R_i) = C_p(O_p(G)) = 1_G$. Thus $P \cap P^x = 1_G$, and we are done. Thus $p$ is a Mersenne prime, and for some $i$, (b) must hold within $PR_i$ (or we can repeat the above argument).

We may assume then that $r_1 = 2$. For $i > 1$, by induction, there is $x_i \in R_i$ with $P \cap P^x_i = C_p(R_i)$. Let $x = x_2 \cdots x_n$. Then as above, $P \cap P^x \leq \bigcap_{i=2}^n C_p(R_i)$. We may assume then that there is some nonidentity element $u$ in $\bigcap_{i=2}^n C_p(R_i)$. Then $O_p(G) = R_iC_{O_p(G)}(u)$, so that $[O_p(G), u] \leq [R_1, u] \leq R_1$, and hence $[O_p(G), u] \leq O_2(G)$.

Now $[O_p(G), u] \neq 1_G$, so $[O_2(G), u] \neq 1_G$, as $[O_p(G), u] = [O_p(G), u, u] \leq [O_2(G), u]$. In particular $P$ does not centralize $O_2(G)$. Now, however, as (b) does not hold in $G$, (b) does not hold within $PR_1$, contrary to assumption.

Hence $n = 1$, so that $O_p(G)$ is a $q$-group for some prime $q \neq p$. Let $R = O_p(G)$. We claim that $R$ is elementary abelian. If not, then $O_p(G/\Phi(R)) = 1_G$, and by induction, there is $\bar{x}$ in $\bar{G} = G/\Phi(R)$ with $\bar{P} \cap \bar{P}^* = 1_{\bar{G}}$. For any preimage $x$ of $\bar{x}$, we have $P \cap P^x = 1_G$, so we may indeed suppose that $R$ is elementary abelian.

Now, as $R$ is elementary abelian, and neither (a) nor (b) hold in $G = PR$, we may appeal to the main theorem of Hargreaves [1] to conclude that for some $y \in R$, $C_p(y) = 1_G$. However, $P \cap P^y \leq C_p(y)$, as $y \in O_p(G)$, so that $P \cap P^y = 1_G$.

The proof of Theorem 1 is complete.

Remark. Of course, conditions (a) and (b) both fail to be satisfied when $G$ has odd order. We also remark that if either (a) or (b) is satisfied, it is possible that $O_p(G)$ can fail to be a Sylow intersection, and the simplest such examples are groups of the form $PQ$, where $P$ is a $p$-group faithfully represented as a group of automorphisms of the elementary abelian $q$-group $Q$ (where the orbit of a $q$ a prime $\neq p$), when $P$ fails to have a regular orbit on $Q$.

We next consider a more general class of groups. First, a definition:

Definition. Let $p$ be an odd prime. Let $\mathcal{S}(p) = \{\text{nonabelian finite simple groups } H \text{ such that for } P \in \text{Syl}_p(\text{Aut}(H)), P \text{ has at least two orbits (by conjugation) on involutions } t \text{ of } H \text{ with } P \cap P^t = 1_H\}$ (where $H$ is considered as a normal subgroup of $\text{Aut}(H)$).

Remark. It seems that most finite simple groups, of order divisible by $p$, are in $\mathcal{S}(p)$. A proof that all (known) simple groups of order divisible by $p$ are in $\mathcal{S}(p)$ seems feasible, but would be extremely tedious.

Before we state and prove Theorem 2, we require a technical lemma. Let $p$ be an odd prime.

Lemma 1. Let $G = PE$, where $P \in \text{Syl}_p(G)$ and $E$ is a minimal normal subgroup of $G$ which is a direct product of nonabelian simple groups, each of which is in $\mathcal{S}(p)$. Then there are involutions $t$ and $u$ of $G$ (which, of course, lie in $E$) such that $P \cap P^t = P \cap P^u = O_p(G)$, and $t$ and $u$ lie in different orbits under conjugation by $P$.

Proof. We proceed by induction on $|G|$. If $E$ is simple, the result follows (upon passage to $G/O_p(G)$) by using the definition of $\mathcal{S}(p)$. Thus we may suppose that $E$
is not simple. Let \( L \) be a simple normal subgroup of \( E \), and let \( S = N_p(L) \). Let \( M \) be a maximal subgroup of \( P \) containing \( S \). (Note that \( S \neq P \) as \( E \) is minimal normal in \( E_P \), but \( E \) is not simple.)

Then for some \( x \in P \setminus M, E = YY^xY^x \cdots Y^{x^{p-1}} \) (where this latter product is direct), where \( Y = L^M \) (a direct product of \( [M:S] \) isomorphic copies of \( L \)).

Let \( a \) and \( b \) be involutions of \( Y \) in different \( M \)-orbits such that \( M \cap M^a = M \cap M^b = O_p(MY) = C_M(Y) \) (by induction, such involutions exist). Let \( z = ab^ix^{p-1}b^x \cdots b^x \) and let \( w = ba^ix^{p-1} \cdots a^x \). Then \( z \) and \( w \) are involutions of \( E \). They are not conjugate under the action of \( P \), for if \( z^{x^m} = w \), where \( 0 \leq i \leq p - 1 \), and \( m \in M \), then if \( i > 1 \), \( b^{x^{im}} = a^x \), so \( b^{x^{i+im-1}} = a \), a contradiction as \( x^p \in M \), but \( a \) and \( b \) are not conjugate under the action of \( M \). If \( i \leq 1 \), we obtain \( b^{x^{i+1}} = a^{x+1} \), again a contradiction.

It remains only to prove that \( P \cap P^z = P \cap P^w = O_p(G) \). We prove that \( P \cap P^z = O_p(G) \) for interchanging the roles of \( a \) and \( b \) then establishes that \( P \cap P^w = O_p(G) \).

Suppose that \( u \in P \cap P^z \). Then \( u^{-1}u^z \in P \), so that \( z^z \in P \cap E \). Let \( u = x^m \), where \( 0 \leq i \leq p - 1 \), and \( m \in M \). Then

\[
(ab^ix^{p-1} \cdots b^x)(a^mb^{x^{-1}m} \cdots ) \in P \cap E.
\]

We first note that \( i = 0 \). If not, then \( b^i a^{x^i}m \in P \cap Y^x \) so \( ba^{x^{im}x^{-i}} \in P \cap Y = M \cap Y \), which forces \( b \) and \( a^{x^{im}x^{-i}} \) to be conjugate by an element of \( M \) (as \( a \) and \( b \) are involutions), a contradiction. Thus \( i = 0 \).

Hence \( (ab^ix^{p-1} \cdots b^x)(a^mb^{x^{-1}m} \cdots b^x) \in P \cap E \), so for \( 1 \leq i \leq p - 1 \), \( b^xb^{x^i} \in P \cap Y^x \), so \( b^{x^m}x^{-i}m \in M \cap Y \). Thus \( x^mx^{-i}m \in M \cap M^b = C_M(Y) \). Also, \( aa^m \in M \cap Y \), so \( m \in M \cap m^a \) and \( m \in C_M(Y) \). Hence \( m \in \cap_{i=0}^{p-1} C_M(Y^x) = C_p(E) = O_p(G) \). Thus we have shown that \( P \cap P^z = O_p(G) \), as claimed.

**Theorem 2.** Let \( G \) be a finite group and \( p \) be an odd prime. Let \( P \in \text{Syl}_p(G) \).

Suppose that \( O_p(G) = 1_G \), and if \( p \) is a Mersenne prime, assume either that \( O_2(G) \leq Z(G) \), or else that \( P \) does not involve \( Z_p \sim Z_p \). Suppose further that for each component, \( L \), of \( G \) such that \( p | |L| \), \( L/Z(L) \) is in \( \mathcal{S}(p) \). Then there is a Sylow \( p \)-subgroup, \( Q \), of \( G \) such that \( P \cap Q = 1_G \).

**Proof.** We proceed by induction on \( |G| \). Since \( C_G(F^*(G)) \leq F^*(G) \) and \( O_p(G) = 1_G \), we see that \( O_p(F^*(G)P) = 1_G \). We may suppose, then, that \( G = PF^*(G) \). If \( O_2(G) \leq Z(G) \), we can suppose that \( F^*(G) = E(G)O(F(G)) \).

Next, we claim that \( Z(E(G)) = 1_G \). For let \( Z = Z(E(G)) \). If \( Z \neq 1_G \), then by induction, for some \( \bar{x} \in \bar{G} (= G/Z) \), we have \( \bar{P} \cap \bar{P}^{\bar{x}} = 1_G \) (for it is easy to check that \( O_p(\bar{G}) = 1_G \)). Then for some preimage, \( x \), say, of \( \bar{x} \), we have \( P \cap P^x = 1_G \).

Now we claim that \( F(G) = 1_G \). If \( F(G) \neq 1_G \), let \( F = F(G) \) and \( E = E(G) \). Since \( Z(E) = 1_G \), \( F^*(G) = F \times E \). By Theorem 1, there is some \( f \in F \) such that \( P \cap P^f = O_p(PF) = C_p(F) \). By induction, there is some \( e \in E \) such that \( P \cap P^e = C_p(E) \).

It is easy to check that \( P \cap P^{ef} \leq C_p(F^E) = 1_G \). Thus \( F = 1 \). A similar argument, using Theorem 1, shows that \( O_p(G) = 1_G \).
If $E$ is not a minimal normal subgroup, but $E = E_1 \times E_2$, where each $E_i \triangleleft EP$, then, by induction for each $i$ there is an $e_i \in E_i$ with $P \cap P^{e_i} = C_p(E_i)$. It quickly follows that $P \cap P^{e_1 e_2} = 1_G$. Thus $E$ is a minimal normal subgroup of $G$. Now we may appeal to Lemma 1 to conclude that for some $e \in E$, we have $P \cap P^e = 1_G$. The proof of Theorem 2 is complete.

REFERENCES