PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 90, Number 1, January 1984

ON SYLOW INTERSECTIONS IN FINITE GROUPS

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ABSTRACT. In general, for a given prime p and finite group G, there need not be Sylow p-subgroups P and Q of G with $P \cap Q = O_p(G)$. In this paper we show that if G is p-soluble, and p is not 2 or a Mersenne prime, then such Sylow p-subgroups exist (also we give conditions guaranteeing the existence of such Sylow subgroups when p is 2 or a Mersenne prime). We also show that if G is not p-soluble, but p is odd and the components of $G/O_p(G)$ are in a certain class of quasi-simple groups, then there are Sylow p-subgroups P and Q of G with $P \cap Q = O_p(G)$, unless perhaps p is a Mersenne prime. When G is p-soluble, our work extends results of N. Itô [**2**].

In general, when G is a finite group and p is a prime divisor of $|G|$, there need not be Sylow p-subgroups P and Q of G such that $P \cap Q = O_p(G)$. The aim of this note is to prove that in many situations, though, there are such Sylow p-subgroups.

Our first observation is that "regular-orbit" theorems such as that of Hargreaves [**1**] can be used to establish the existence of such Sylow subgroups in p-solvable groups when $p \neq 2$ or a Mersenne prime. More precisely, we can prove the following theorem, which slightly strengthens a result of Itô [**2**].

THEOREM 1. *Let G be a finite p-solvable group, and $P \in \mathrm{Syl}p(G)$. Then there is a Sylow p-subgroup, Q, of G with $P \cap Q = O_p(G)$, unless, perhaps, one of the following situations occur*:

(a) *$p = 2$, $P/O_p(G)$ involves $Z_2 \sim Z_2$ and $|O_{p,p'}(G)|$ is divisible by the square of a Fermat prime, or the square of a Mersenne prime.*

(b) *p is a Mersenne prime, $P/O_p(G)$ involves $Z_p \sim Z_p$, $P/O_p(G)$ does not centralize $O_2(G/O_p(G))$, and $(p+1)^p$ divides $|G|$.*

PROOF. We proceed by induction on $|G|$. Suppose that neither (a) nor (b) hold. Certainly we may suppose that $O_p(G) = 1_G$. By the Hall-Higman centralizer lemma, $C_G(O_{p'}(G)) \leqslant O_{p'}(G)$. Thus $O_p(PO_{p'}(G)) = 1_G$, so we may suppose that $G = PO_{p'}(G)$.

We claim that $O_{p'}(G)$ is a q-group for some prime $q \neq p$. Let $\{r_i\colon 1 \leqslant i \leqslant n\}$ be the set of prime divisors of $|O_{p'}(G)|$, and for each i, let R_i be a P-invariant Sylow r_i-subgroup of $O_{p'}(G)$. Suppose that $n > 1$. If p is not a Mersenne prime, then neither (a) nor (b) hold in PR_i, so by induction there are some x_i in R_i with $P \cap P^{x_i} = C_p(R_i)$. Let $x = x_1x_2 \cdots x_n$. Then x centralizes $P \cap P^x$. Let $u \in P \cap P^x$. Then $x^u = x$, so that $x_1^u x_2^u \cdots x_n^u = x_1x_2 \cdots x_n$. Now $x_i^u \in R_i$ for each i, and

Received by the editors March 10, 1983.
1980 *Mathematics Subject Classification*. Primary 20D20; Secondary 20D99.

each element of $O_{p'}(G)$ has a *unique* expression of the form $z_1 z_2 \cdots z_n$, where each $z_i \in R_i$. Thus $x_i^u = x_i$ for each i, so that $u \in P \cap P^{x_i}$ for each i. Consequently, $u \in \bigcap_{i=1}^n C_p(R_i) = C_p(O_{p'}(G)) = 1_G$. Thus $P \cap P^x = 1_G$, and we are done. Thus p is a Mersenne prime, and for some i, (b) must hold within PR_i (or we can repeat the above argument).

We may assume then that $r_1 = 2$. For $i > 1$, by induction, there is $x_i \in R_i$ with $P \cap P^{x_i} = C_p(R_i)$. Let $x = x_2 \cdots x_n$. Then as above, $P \cap P^x \leqslant \bigcap_{i=2}^n C_p(R_i)$. We may assume then that there is some nonidentity element u in $\bigcap_{i=2}^n C_p(R_i)$. Then $O_{p'}(G) = R_1 C_{O_{p'}(G)}(u)$, so that $[O_{p'}(G), u] \leqslant [R_1, u] \leqslant R_1$, and hence $[O_{p'}(G), u] \leqslant O_2(G)$.

Now $[O_{p'}(G), u] \neq 1_G$, so $[O_2(G), u] \neq 1_G$, as $[O_{p'}(G), u] = [O_{p'}(G), u, u] \leqslant [O_2(G), u]$. In particular P does not centralize $O_2(G)$. Now, however, as (b) does not hold in G, (b) does not hold within PR_1, contrary to assumption.

Hence $n = 1$, so that $O_{p'}(G)$ is a q-group for some prime $q \neq p$. Let $R = O_{p'}(G)$. We claim that R is elementary abelian. If not, then $O_p(G/\Phi(R)) = 1_G$, and by induction, there is $\bar{x}$ in $\bar{G} = G/\Phi(R)$ with $\bar{P} \cap \bar{P}^{\bar{x}} = 1_{\bar{G}}$. For any preimage x of $\bar{x}$, we have $P \cap P^x = 1_G$, so we may indeed suppose that R is elementary abelian.

Now, as R is elementary abelian, and neither (a) nor (b) hold in $G = PR$, we may appeal to the main theorem of Hargreaves [**1**] to conclude that for some $y \in R$, $C_p(y) = 1_G$. However, $P \cap P^y \leqslant C_p(y)$, as $y \in O_{p'}(G)$, so that $P \cap P^y = 1_G$.

The proof of Theorem 1 is complete.

REMARK. Of course, conditions (a) and (b) both fail to be satisfied when G has odd order. We also remark that if either (a) or (b) is satisfied, it is possible that $O_p(G)$ can fail to be a Sylow intersection, and the simplest such examples are groups of the form PQ, where P is a p-group faithfully represented as a group of automorphisms of the elementary abelian q-group Q (q a prime $\neq p$), when P fails to have a regular orbit on Q.

We next consider a more general class of groups. First, a definition:

DEFINITION. Let p be an odd prime. Let $\mathfrak{S}(p) = \{$nonabelian finite simple groups H such that for $P \in \mathrm{Sylp}(\mathrm{Aut}(H))$, P has at least two orbits (by conjugation) on involutions t of H with $P \cap P^t = 1_H\}$ (where H is considered as a normal subgroup of $\mathrm{Aut}(H)$).

REMARK. It seems that most finite simple groups, of order divisible by p, are in $\mathfrak{S}(p)$. A proof that all (known) simple groups of order divisible by p are in $\mathfrak{S}(p)$ seems feasible, but would be extremely tedious.

Before we state and prove Theorem 2, we require a technical lemma. Let p be an odd prime.

LEMMA 1. *Let $G = PE$, where $P \in \mathrm{Sylp}(G)$ and E is a minimal normal subgroup of G which is a direct product of nonabelian simple groups, each of which is in $\mathfrak{S}(p)$. Then there are involutions t and u of G (which, of course, lie in E) such that $P \cap P^t = P \cap P^u = O_p(G)$, and t and u lie in different orbits under conjugation by P.*

PROOF. We proceed by induction on $|G|$. If E is simple, the result follows (upon passage to $G/O_p(G)$) by using the definition of $\mathfrak{S}(p)$. Thus we may suppose that E

is not simple. Let L be a simple normal subgroup of E, and let $S = N_p(L)$. Let M be a maximal subgroup of P containing S. (Note that $S \neq P$ as E is minimal normal in EP, but E is not simple.)

Then for some $x \in P \setminus M$, $E = YY^xY^{x^2} \cdots Y^{x^{p-1}}$ (where this latter product is direct), where $Y = L^M$ (a direct product of $[M:S]$ isomorphic copies of L).

Let a and b be involutions of Y in different M-orbits such that $M \cap M^a = M \cap M^b = O_p(MY) = C_M(Y)$ (by induction, such involutions exist). Let $z = ab^xb^{x^2} \cdots b^{x^{p-1}}$ and let $w = ba^xa^{x^2} \cdots a^{x^{p-1}}$. Then z and w are involutions of E. They are not conjugate under the action of P, for if $z^{x^i m} = w$, where $0 \leqslant i \leqslant p-1$, and $m \in M$, then if $i > 1$, $b^{x^{p+1}m} = a^x$, so $b^{x^{p(xmx^{-1})}} = a$, a contradiction as $x^p \in M$, but a and b are not conjugate under the action of M. If $i \leqslant 1$, we obtain $b^{x^{i+1}m} = a^{x^{i+1}}$, again a contradiction.

It remains only to prove that $P \cap P^z = P \cap P^w = O_p(G)$. We prove that $P \cap P^z = O_p(G)$ for interchanging the roles of a and b then establishes that $P \cap P^w = O_p(G)$.

Suppose that $u \in P \cap P^z$. Then $u^{-1}u^z \in P$, so that $zz^u \in P \cap E$. Let $u = x^i m$, where $0 \leqslant i \leqslant p-1$, and $m \in M$. Then

$$\left(ab^xb^{x^2} \cdots b^{x^{p-1}}\right)\left(a^{x^i m}b^{x^{i+1}m} \cdots\right) \in P \cap E.$$

We first note that $i = 0$. If not, then $b^{x^i}a^{x^i m} \in P \cap Y^{x^i}$ so $ba^{x^i m x^{-i}} \in P \cap Y = M \cap Y$, which forces b and $a^{x^i m x^{-i}}$ to be conjugate by an element of M (as a and b are involutions), a contradiction. Thus $i = 0$.

Hence $(ab^x \cdots b^{x^{p-1}})(a^m b^{xm} \cdots b^{x^{p-1}m}) \in P \cap E$, so for $1 \leqslant i \leqslant p-1$, $b^{x^i}b^{x^i m} \in P \cap Y^{x^i}$, so $bb^{x^i m x^{-i}} \in M \cap Y$. Thus $x^i m x^{-i} \in M \cap M^b = C_M(Y)$. Also, $aa^m \in M \cap Y$, so $m \in M \cap M^a$ and $m \in C_M(Y)$. Hence $m \in \bigcap_{i=0}^{p-1} C_M(Y^{x^i}) = C_p(E) = O_p(G)$. Thus we have shown that $P \cap P^z = O_p(G)$, as claimed.

THEOREM 2. *Let G be a finite group and p be an odd prime. Let $P \in$* Sylp(G). *Suppose that $O_p(G) = 1_G$, and if p is a Mersenne prime, assume either that $O_2(G) \leqslant Z(G)$, or else that P does not involve $Z_p \sim Z_p$. Suppose further that for each component, L, of G such that $p \mid |L|$, $L/Z(L)$ is in $\mathcal{S}(p)$. Then there is a Sylow p-subgroup, Q, of G such that $P \cap Q = 1_G$.*

PROOF. We proceed by induction on $|G|$. Since $C_G(F^*(G)) \leqslant F^*(G)$ and $O_p(G) = 1_G$, we see that $O_p(F^*(G)P) = 1_G$. We may suppose, then, that $G = PF^*(G)$. If $O_2(G) \leqslant Z(G)$, we can suppose that $F^*(G) = E(G)O(F(G))$.

Next, we claim that $Z(E(G)) = 1_G$. For let $Z = Z(E(G))$. If $Z \neq 1_G$, then by induction, for some $\bar{x} \in \bar{G} (= G/Z)$, we have $\bar{P} \cap \bar{P}^{\bar{x}} = 1_{\bar{G}}$ (for it is easy to check that $O_p(\bar{G}) = 1_{\bar{G}}$). Then for some preimage, x say, of $\bar{x}$, we have $P \cap P^x = 1_G$.

Now we claim that $F(G) = 1_G$. If $F(G) \neq 1_G$, let $F = F(G)$ and $E = E(G)$. Since $Z(E) = 1_G$, $F^*(G) = F \times E$. By Theorem 1, there is some $f \in F$ such that $P \cap P^f = O_p(PF) = C_p(F)$. By induction, there is some $e \in E$ such that $P \cap P^e = C_p(E)$. It is easy to check that $P \cap P^{ef} \subseteq C_p(EF) = 1_G$. Thus $F = 1$. A similar argument, using Theorem 1, shows that $O_{p'}(G) = 1_G$.

If E is not a minimal normal subgroup, but $E = E_1 \times E_2$, where each $E_i \lhd EP$, then, by induction for each i there is an $e_i \in E_i$ with $P \cap P^{e_i} = C_p(E_i)$. It quickly follows that $P \cap P^{e_1 e_2} = 1_G$. Thus E is a minimal normal subgroup of G. Now we may appeal to Lemma 1 to conclude that for some $e \in E$, we have $P \cap P^e = 1_G$. The proof of Theorem 2 is complete.

References

1. B. B. Hargreaves, *The existence of regular orbits for nilpotent groups*, J. Algebra **72** (1981), 54–100.
2. N. Itô, *Über den Kleinsten p-Durchschnitt ausflösbarer Gruppen*, Arch. Math. (Basel) **9** (1958), 27–32.

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