RELATIVE BARR-RINEHART
AND COTRIPLE COHOMOLOGY GROUPS
ARE ISOMORPHIC

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Abstract. The theorem, stated in the title of this article, is proved.

Several people have asked about the relationship between the relative cohomology 
groups defined by Barr and Rinehart [3], and those of [6]. The point of this note is to 
show that these groups (in the cases considered by Barr and Rinehart) coincide.

In a sense, this result adds perspective on the creative tension which exists 
between derived functors and Ext groups. Barr has examples (see, e.g. [1]) which 
show that, in the theory of commutative rings, the functor which classifies extensions 
is definitely not a derived functor. By the results of Beck [4] and Duskin [5] (see also 
[6]) any algebraic theory defines a cohomology theory, the "cotriple-derived func-
tors", which classifies extensions. The paper of Barr and Rinehart can thus be 
viewed as a statement that for certain theories these intrinsically defined groups can 
equally well be computed by the more classical approach of "ordinary" derived 
functors on a module category. This note shows that the Barr-Rinehart relative 
groups Ext_\k(S/S^2, M) also appear naturally in the cotriple theory. This raises the 
following unanswered question: what conditions on an algebraic theory make it 
possible to compute the intrinsic long exact cohomology sequences via ordinary 
derived functors? Here we shall give conditions, but they are not really conditions on 
the theory.

Let us quickly reassemble the relevant data from [3 and 6] and set forth our 
assumptions. Let U: C -> B, F: B -> C be a tripleable adjoint pair and (G = FU, \epsilon, \delta) 
the corresponding cotriple on C. Let H^*(-, M) be the G-derived functors, with M 
an abelian group in C. Suppose B (and hence C) has finite inverse limits, and let \phi: 
T -> R be a U-split epimorphism in C. Let k^0, k^1: P -> T be a kernel pair of \phi and 
k^0, k^1, k^2: K -> P be a kernel triple of (k^0, k^1). Let

\text{Hom}_{-1}(k^0, k^1; M) = \{ f: G^{n+1}P \to M | f \circ G^{n+1}k^0 + f \circ G^{n+1}k^2 = f \circ G^{n+1}k^1 \}

for n \geq -1, and let Ab(X) denote the category of abelian group objects in category 
X. Then Hom_{-1}(k^0, k^1; -): Ab(C) -> Ab(Sets) is a functor; suppose that

(1) \text{Hom}_{-1}(k^0, k^1; -) \text{ is representable by } S/S^2 \text{ in } Ab(C).

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Caution: $S/S^2$ is merely notation for the representing object and will not in general "be" $S$ modulo its square; the notation is that of Barr and Rinehart. The groups $	ext{Hom}_S(k^0, k^1; M)$ define a subcomplex of $C(G^{+1}P, M)$ and we denote its $m$th homology group by $H^m(\phi; M)$.

Let $\mathcal{Q}$ be the class of all monomorphisms in $Ab(C)$ which are kernels of $U$-split epimorphisms in $Ab(C)$; here $U$ is also being used to denote the functor $Ab(C) \to Ab(B)$ induced by $U: C \to B$. Note that Barr and Rinehart denote by $\mathcal{Q}$ the class of all monomorphisms in $Ab(C)$ which are $U$-split in $Ab(B)$, but in their examples the two definitions yield the same class. An object $M$ in $Ab(C)$ is $\mathcal{Q}$-injective if $Ab(C)(N', M) \to Ab(C)(N, M)$ is surjective whenever $N \to N'$ is in $\mathcal{Q}$. We suppose

\begin{equation}
\text{(2) } Ab(C) \text{ has enough } \mathcal{Q}\text{-injectives, and}
\end{equation}

\begin{equation}
\text{(3) } H^n(\phi; M) = 0 \text{ whenever } n \geq 1 \text{ and } M \text{ is } \mathcal{Q}\text{-injective.}
\end{equation}

With the technicalities now behind us, we proceed to the main result. The following consequence of our definition of $\mathcal{Q}$ is vital.

**LEMMA.** $U: Ab(C) \to Ab(B)$ preserves quotients of members of $\mathcal{Q}$; in fact they are split in $Ab(B)$.

**Proof.** Let $f: A' \to A$ be in $\mathcal{Q}$, and $p: A \to C$ the $U$-split epimorphism in $Ab(C)$ of which it is the kernel. Then any quotient of $f$ in $Ab(C)$ is uniquely isomorphic to $p$, so we need that $Up$ is a quotient of $Uf$ in $Ab(B)$. But $UA' \Rightarrow UA \to UC$ is a split coequalizer and hence we are finished. Q.E.D.

**THEOREM.** For $n \geq 0$, $H^*(\phi; -)$ and $\text{Ext}_n^*(S/S^2, -)$ are isomorphic functors $Ab(C) \to Ab(Sets)$; here $\text{Ext}^n$ is the $n$th derived functor of $Ab(C)(S/S^2, -)$ using the injective class $\mathcal{Q}$ as in [3].

**Proof.** It is only necessary (thanks to assumption (2) above) to prove that $H^*(\phi; -)$ is a $\mathcal{Q}$-exact cohomological $\delta$-functor augmented over $Ab(C)(S/S^2, -)$. By assumption (1), $\text{Hom}_C(k^0, k^1; -) \simeq Ab(C)(S/S^2, -)$; but by 3.5 of [6], $0 \to \text{Hom}_C(k^0, k^1; -) \to \text{Hom}_C(k^0, k^1; -) \to \text{Hom}_C(k^0, k^1; -)$ is exact, so $\text{Hom}_C(k^0, k^1; -) \simeq H^0(\phi; -)$ and hence $Ab(C)(S/S^2, -) \simeq H^0(\phi; -)$. Next suppose $M$ is $\mathcal{Q}$-injective and $n \geq 1$; then by 3.7 of [6], $H^n(T, M) \to H^n(\phi; M) \to H^{n+1}(R, M)$ is exact. Because of assumption (3), $H^n(T, M) = H^{n+1}(R, M) = 0$ and thus $H^n(\phi; M) = 0$. Finally, let $0 \to M' \to M \to M'' \to 0$ be a $\mathcal{Q}$-exact sequence in $Ab(C)$; by the lemma, if $n > 0$ and $X$ is in $\mathcal{C}$ then $0 \to C(G^{+1}X, M') \to C(G^{+1}X, M) \to C(G^{+1}X, M'') \to 0$ is split exact in $Ab(Sets)$, since it is naturally isomorphic to $0 \to B(U^nX, UM') \to B(U^nX, UM) \to B(U^nX, UM'') \to 0$. Since $\phi: T \to R$ is a $U$-split epimorphism, $C(G^{+1}R, N) \to C(G^{+1}T, N)$ is a split monomorphism in $Ab(Sets)$ for any $N$ in $Ab(C)$; recall that its quotient is $\text{Hom}_C(k^0, k^1; N)$. A simple nine-lemma type argument now shows that $0 \to \text{Hom}_C(k^0, k^1, M') \to \text{Hom}_C(k^0, k^1, M) \to \text{Hom}_C(k^0, k^1, M'') \to 0$ is exact for $n > 0$. From the resulting short exact sequence of cochain complexes we get a long exact sequence $\cdots \to H^n(\phi; M) \to H^n(\phi; M') \to H^{n+1}(\phi; M') \to H^{n+1}(\phi; M) \to \cdots$, and the connecting homomorphisms are natural. Q.E.D.
It remains to show how to use this theorem to verify that the relative cotriple groups $H^n(\phi; M)$ are isomorphic to the Barr-Rinehart groups $\text{Ext}^n_R(S/S^2, M)$. Let $\mathcal{C}$ and $\mathcal{C}_0$ be as in [3], with $U'' : \mathcal{C} \to \mathcal{C}_0$ the forgetful functor and $F' : \mathcal{C}_0 \to \mathcal{C}$ its left adjoint (as described in [3]). For $R$ in $\mathcal{C}$ let $C = \mathcal{C}/R$, $B = \mathcal{C}_0/U'R$ and $U : C \to B$, $F : B \to C$ the extensions of $U'$, $F$ to these comma categories (e.g. $F(S \to U'R) = F'S \to F'U'R \to R$). It is by now well known that $U'$ and hence $U$ are tripleable in each of the cases $A$, $S$, $L$, and $G$ of [3] (which are, respectively, unitary $K$-algebras, supplemented unitary $K$-algebras, Lie algebras over $K$, and groups), and that the $H^n(X, M)$ defined by Barr and Rinehart agree with the resulting cotriple cohomology groups (see e.g. [2]). Thus assumption (3) above is true. Much as in §7 of [6], one can show that $S/S^2$ as defined by Barr and Rinehart represents $\text{Hom}_\mathcal{C}(k^0, k^1; -)$ in cases $A$, $S$, $L$, and $G$. This is assumption (1) above. Finally, assumption (2) is proved in [3].

**REFERENCES**


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