CO-WELL-POWERED REFLECTIVE SUBCATEGORIES

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Abstract. A full isomorphism-closed subcategory $\mathcal{A}$ of a complete well-powered and co-well-powered category $\mathcal{C}$ is both co-well-powered (in its own right) and reflective in $\mathcal{C}$ if and only if

(a) $\mathcal{A}$ is closed in $\mathcal{C}$ under the formation of $(U$-small-indexed) limits, and

(b) the epi-reflective hull $\mathcal{B}$ of $\mathcal{A}$ in $\mathcal{C}$ is co-well-powered.

A full isomorphism-closed subcategory $\mathcal{Y}$ of a well-powered and co-well-powered complete category $\mathcal{X}$ is epi-reflective in $\mathcal{X}$ (i.e. full, isomorphism-closed and reflective with epimorphic reflection morphisms) if and only if $\mathcal{Y}$ is stable in $\mathcal{X}$ under the formation of products (indexed over $U$-small sets) and under extremal subobjects—where $U$ denotes a fixed universe (cf. [2, p. 87; 6, p. 1276; 7, p. 356]). Indeed, a slightly weaker requirement suffices to ensure the epi-reflectiveness of $\mathcal{Y}$ in $\mathcal{X}$: $\mathcal{Y}$ is closed under $(U$-small-indexed) products and strongly closed under difference kernels, i.e. whenever $A \rightarrow B \xrightarrow{f} C$ is a difference kernel (or equalizer) in $\mathcal{X}$ with $B \in \text{Ob} \mathcal{Y}$, then $A \in \text{Ob} \mathcal{Y}$ [3, 10.2.1].

The smallest epi-reflective subcategory $\mathcal{D}$ of a well-powered and co-well-powered complete category $\mathcal{C}$ containing a given subcategory $\mathcal{X}$ of $\mathcal{C}$, the "epi-reflective hull" $\mathcal{D}$ of $\mathcal{X}$ in $\mathcal{C}$, consists of all $\mathcal{C}$-objects which are (domains of) extremal subobjects of products (over a $U$-small index set) of members of Ob $\mathcal{X}$. Every full isomorphism-closed reflective subcategory $\mathcal{A}$ of a well-powered and co-well-powered complete category $\mathcal{C}$ is both mono-reflective and (consequently) epi-reflective in the epi-reflective hull $\mathcal{B}$ of $\mathcal{A}$ in $\mathcal{C}$ [1]. Sharpening results in [1 and 7], it is observed in [4] that a full subcategory $\mathcal{A}$ of a complete, well-powered and co-well-powered category $\mathcal{C}$ is reflective in $\mathcal{C}$ if (i) and (ii) are satisfied:

(i) $\mathcal{A}$ is stable in $\mathcal{C}$ under the formation of $(U$-small-indexed) limits;

(ii) the epi-reflective hull $\mathcal{B}$ of $\mathcal{A}$ in $\mathcal{C}$ is co-well-powered.

(Indeed, a difference kernel $X \xrightarrow{u} Y \xrightarrow{v} Z$ in $\mathcal{B}$ with $Y \in \text{Ob} \mathcal{A}$ yields a difference kernel $X \xrightarrow{u} Y \xrightarrow{m} Z$ for some extremal monomorphism $m: Z \rightarrow A$ in $\mathcal{C}$ with $A \in \text{Ob} \mathcal{A}$; hence $X \in \text{Ob} \mathcal{A}$ by hypothesis. Consequently, $\mathcal{A}$ is epi-reflective in $\mathcal{B}$.)

While (i) is clearly necessary for reflectiveness of $\mathcal{A}$ in $\mathcal{C}$, condition (ii) is not. Here we wish to add the following observations.
1. A full subcategory \( \mathcal{A} \) of a complete, well-powered and co-well-powered category \( \mathcal{C} \) is co-well-powered if the epi-reflective hull \( \mathcal{B} \) of \( \mathcal{A} \) in \( \mathcal{C} \) is also.

   Indeed, the inclusion \( \mathcal{A} \to \mathcal{B} \) preserves epimorphisms [5, 2.2].

2. A full isomorphism-closed subcategory \( \mathcal{A} \) of a complete well-powered and co-well-powered category \( \mathcal{C} \) is both co-well-powered (in its own right) and reflective in \( \mathcal{C} \) if and only if

   (a) \( \mathcal{A} \) is closed in \( \mathcal{C} \) under the formation of (\( U \)-small-indexed) limits, and

   (b) the epi-reflective hull \( \mathcal{B} \) of \( \mathcal{A} \) in \( \mathcal{C} \) is co-well-powered.

   \textbf{Proof.} It remains to establish the necessity of (b). For \( X \in \text{Ob} \mathcal{C} \), let \( r_X : X \to R(X) \) denote the \( \mathcal{A} \)-reflection morphism of \( X \). Now let \( B \in \text{Ob} \mathcal{B} \). For every \( B \)-epimorphism \( e : B \to Y \) we obtain a commutative square:

\[
\begin{array}{ccc}
R(B) & \xrightarrow{R(e)} & R(Y) \\
\uparrow r_B & & \uparrow r_Y \\
B & \xrightarrow{e} & Y
\end{array}
\]

Since \( r_Y \) is a \( B \)-epimorphism, so is \( r_Y e = R(e)r_B \); hence so is \( R(e) \). Consequently, \( R(e) \) is an epimorphism in \( \mathcal{A} \). There are suitable isomorphisms \( j_e \) in \( \mathcal{A} \) with domain \( R(Y) \) so that we obtain a mapping \( \varphi : e \mapsto j_e R(e) \) from a representative system of \( B \)-epimorphisms with domain \( B \) into a representative system of \( \mathcal{A} \)-epimorphisms with domain \( R(B) \). The latter set is \( U \)-small by hypothesis. Since \( r_Y \) is an (extremal) monomorphism and since \( \mathcal{C} \) is well-powered, the fibers of this mapping \( \varphi \) (i.e. the inverse images of single elements) are \( U \)-small. As a consequence, the domain of \( \varphi \) is also \( U \)-small, i.e. \( \mathcal{B} \) is co-well-powered.

   Necessary and sufficient conditions for a subcategory to be co-well-powered reflective are also given in [1, Theorem 3].

\textbf{Note added in proof.} A careful examination of the proofs of the results leading to the theorem obtained above shows that the latter can be extended to a complete category \( \mathcal{C} \) with a “well-founded” bicategory structure \((E, M)\) [7, p. 355] when the epi-reflective hull \( \mathcal{B} \) of \( \mathcal{A} \) is replaced by the \( E \)-reflective hull of \( \mathcal{A} \) in \( \mathcal{C} \).

   (This transfers co-well-poweredness from compact \( T_2 \)-spaces to completely Hausdorff spaces (with continuous maps) as well as to every intermediate full subcategory.)

\textbf{References}


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