PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 90, Number 1, January 1984

OSCILLATION THEOREMS FOR nTH ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

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ABSTRACT. In this note we study the oscillatory behavior of solutions of the nth order nonlinear functional differential equation

$$x^{(n)}(t) + q(t)f(x[g(t)]) = 0, \qquad n \text{ even},$$

without assuming that the deviating argument is retarded or advanced. Sufficient conditions are established for all solutions of the equation to be oscillatory.

The purpose of this note is to obtain some oscillation criteria for the nth order equation

$$x^{(n)}(t) + q(t)f(x[g(t)]) = 0, \qquad n \text{ even}, \tag{1}$$

where the following conditions are assumed to hold:

(2) $q, g \in C[[t_0, \infty), R]$, $q(t)$ is nonnegative and not identically zero on any interval of the form $[t_1, \infty)$, $t_1 \geqslant t_0$ and $\lim_{t\to\infty} g(t) = \infty$;

(3) $f \in C[R, R]$, f is nondecreasing $0 < f(x) \leqslant -f(-x)$ for $x > 0$;

(4) $\int_0^{\alpha} du/f(u) < \infty$ and $\int_{-0}^{-\alpha} du/f(u) < \infty$ for any $\alpha > 0$;

(5) $f(xy) \geqslant Kf(x)f(y)$ for $x, y > 0$ and K is a positive constant.

In what follows, by a proper solution of (1), we mean a function $x \in C^n[[T_x, \infty), R]$ which satisfies (1) for all sufficiently large t and $\sup\{|x(t)| : t \geqslant T\} > 0$ for any $T \geqslant T_x$. A proper solution of (1) is called oscillatory if it has arbitrarily large zeros and it is called nonoscillatory otherwise.

We will have an occasion to use the following two lemmas given in [**1**].

LEMMA 1. *Let u be a positive and n-times differentiable function on an interval $[t_0, \infty)$. If $u^{(n)}$ is of constant sign and not identically zero on any interval of the form $[t_1, \infty)$, $t_1 \geqslant t_0$, then there exists a $t_u \geqslant t_0$ and an integer l, $0 \leqslant l \leqslant n$ with $n + l$ even for $u^{(n)} \geqslant 0$ or $n + l$ odd for $u^{(n)} \leqslant 0$ and such that $l > 0$ implies $u^{(k)}(t) > 0$ for every $t \geqslant t_u$ $(k = 0, 1, \ldots, l - 1)$ and $l \leqslant n - 1$ implies $(-1)^{l+k}u^{(k)}(t) > 0$ for every $t \geqslant t_u$ $(k = l, l + 1, \ldots, n - 1)$.*

LEMMA 2. *If the function u is as in Lemma 1 and*

$$u^{(n-1)}(t)u^{(n)}(t) \leqslant 0 \quad \text{for every } t \leqslant t_u,$$

Received by the editors January 18, 1982 and, in revised form, September 8, 1982.
1980 *Mathematics Subject Classification*. Primary 34K25.

0002-9939/84 $1.00 + $.25 per page

then for every λ, $0 < \lambda < 1$, *there exists an* $M_\lambda > 0$ *such that*

$$u[\lambda t] \geqslant M_\lambda t^{n-1}|u^{(n-1)}(t)| \quad \textit{for all large } t.$$

Moreover, if u is increasing, then

$$u(t) \geqslant u[\tfrac{1}{2}t] \geqslant M_{1/2}t^{n-1}|u^{(n-1)}(t)| \quad \textit{for all large } t.$$

REMARK. From the proof of Lemma 2 [**1**, Lemma 2], we can take

$$M_{1/2} = \frac{2^{2-2^n}}{(n-1)!}.$$

We also need the following lemma.

LEMMA 3. *Let* $n = 1$, *and conditions* (2)–(4) *hold. If*

$$\int_A q(s)\,ds = \infty,$$

where $A = \{t \in [t_0, \infty)\colon t_0 \leqslant g(t) < t\}$, *the retarded part of* $g(t)$, *then all proper solutions of* (1) *are oscillatory.*

PROOF. The proof is similar to that of Theorem 2 in [**3**] and hence is omitted.

THEOREM 1. *Let conditions* (2)–(5) *hold. Then all proper solutions of* (1) *are oscillatory if*

$$\int_A q(s)f(g^{n-1}(s))\,ds = \infty, \tag{6}$$

where $A = \{t \in [t_0, \infty)\colon t_0 \leqslant g(t) < t\}$, *the retarded part of* $g(t)$.

PROOF. Let $x(t)$ be a nonoscillatory solution of (1), say $x(t) > 0$ for $t \geqslant t_1 \geqslant t_0$. Then there exists $t_2 \geqslant t_1$ such that $x[g(t)] > 0$ for $t \geqslant t_2$.

By Lemma 1, there exists a $t_3 \geqslant t_2$ such that

$$\dot{x}(t) > 0 \quad \text{and} \quad x^{(n-1)}(t) > 0 \quad \text{for } t \geqslant t_3. \tag{7}$$

Next, applying Lemma 2 for $u = x$, we derive that there exists a $t_4 \geqslant t_3$ such that

$$x(t) \geqslant M_{1/2}t^{n-1}x^{(n-1)}(t) \quad \text{for } t \geqslant t_4.$$

Consequently, since $g(t) \to \infty$ as $t \to \infty$, there exists a $t_5 \geqslant t_4$ such that

$$x[g(t)] \geqslant M_{1/2}g^{n-1}(t)x^{(n-1)}[g(t)] \quad \text{for } t \geqslant t_5. \tag{8}$$

Thus (1) becomes

$$x^{(n)}(t) + q(t)f\big(M_{1/2}g^{n-1}(t)x^{(n-1)}[g(t)]\big) \leqslant 0 \quad \text{for } t \geqslant t_5.$$

Now, using (5) we get

$$x^{(n)}(t) + K^2q(t)f(M_{1/2})f(g^{n-1}(t))f(x^{(n-1)}[g(t)]) \leqslant 0$$

or

$$x^{(n)}(t) + cq(t)f(g^{n-1}(t))f(x^{(n-1)}[g(t)]) \leqslant 0 \tag{9}$$

for some constant $c > 0$. Now if we let $u(t) = x^{(n-1)}(t)$, then (9) reduces to

$$\dot{u}(t) + cq(t)f(g^{n-1}(t))f(u[g(t)]) \leqslant 0.$$

Using Lemma 3, we obtain the desired contradiction.

If $x(t) < 0$ for $t \geqslant t_1 \geqslant t_0$, then the transformation $y = -x$ transforms (1) to

$$(1') \qquad y^{(n)}(t) + q(t)f^*(y[g(t)]) = 0, \qquad n \text{ even},$$

where $f^*(y) = -f(-y)$. Thus the above argument can be repeated on the positive solutions $y(t)$ of (1′); this in turn yields the required results for negative solutions of (1).

For illustration consider the following example.

EXAMPLE 1. Consider the equation

$$(10) \qquad x^{(6)}(t) + (t-1)^{-8/3}x^{1/3}[t + \sin t] = 0, \qquad t \geqslant \pi.$$

The retarded part of $g(t) = t + \sin t$ is $A = \bigcup_{k=1}^{\infty}((2k-1)\pi, 2k\pi)$.

$$\begin{aligned}\int_A q(s)f(g^{n-1}(s))\,ds &= \sum_{k=1}^{\infty}\int_{(2k-1)\pi}^{2k\pi}\frac{(t+\sin t)^{5/3}}{(t-1)^{8/3}}\,dt\\ &\geqslant \sum_{k=1}^{\infty}\int_{(2k-1)\pi}^{2k\pi}\frac{(t-1)^{5/3}}{(t-1)^{8/3}}\,dt\\ &= \sum_{k=1}^{\infty}\log\frac{2k\pi-1}{2k\pi-\pi-1} = \infty,\end{aligned}$$

and so from Theorem 1, it follows that all proper solutions of (1) are oscillatory.

The following theorem extends and improves Theorem 3.2 of Wong [**9**] who considered the equation

$$(11) \qquad \ddot{x}(t) + q(t)x[g(t)] = 0,$$

where $ct \leqslant g(t) \leqslant t$, $c > 0$ and $q(t) \geqslant 0$.

THEOREM 2. *Let conditions* (2) *and* (3) *hold, and*

$$(12) \qquad f'(x) > \alpha > 0 \quad \text{for } x \neq 0 \; (' = d/dt).$$

Suppose further that there exist functions $\rho, \sigma \in C^1[[t_0, \infty), (0, \infty)]$ *such that*

$$(13) \quad \sigma(t) \leqslant \min\{t, 2g(t)\},\ \dot{\sigma}(t) > 0 \text{ and } \sigma(t) \to \infty \text{ as } t \to \infty \; (\cdot = d/dt);$$

and

$$(14) \qquad \limsup_{t\to\infty}\int_{t_0}^{t}\left[\rho(s)q(s) - \frac{\dot{\rho}^2(s)}{2\alpha M_{1/2}\sigma^{n-2}(s)\dot{\sigma}(s)\rho(s)}\right]ds = \infty,$$

where $M_{1/2} = 2^{2-2n}/(n-1)!$ *(*$M_{1/2}$ *is as in Lemma* 2 *for* $\lambda = \frac{1}{2}$*). Then every solution of equation* (1) *is oscillatory.*

PROOF. Let $x(t)$ be a nonoscillatory solution of (1), say $x(t) > 0$ for $t \geqslant t_1 \geqslant t_0$. It follows, as in the proof of Theorem 1 that there exists a $t_3 \geqslant t_2$ so that

$$\dot{x}(t) > 0 \quad \text{and} \quad x^{(n-1)}(t) > 0 \quad \text{for } t \geqslant t_3.$$

Now, applying Lemma 2 for $u = \dot{x}$ and $\lambda = \frac{1}{2}$, we easily derive that there exists $t_4 \geqslant t_3$ such that

$$\dot{x}\left[\tfrac{1}{2}t\right] \geqslant M_{1/2}t^{n-2}x^{(n-1)}(t) \quad \text{for } t \geqslant t_4.$$

Since $\sigma(t) \to \infty$ as $t \to \infty$, there exists $t_5 \geqslant t_4$ such that

$$\dot{x}\left[\tfrac{1}{2}\sigma(t)\right] \geqslant M_{1/2}\sigma^{n-2}(t)x^{(n-1)}[\sigma(t)] \quad \text{for } t \geqslant t_5.$$

Now, by condition (13) and the fact that $x^{(n-1)}(t)$ is nonincreasing, we have

$$\dot{x}\left[\tfrac{1}{2}\sigma(t)\right] \geqslant M_{1/2}\sigma^{n-2}(t)x^{(n-1)}(t) \quad \text{for } t \geqslant t_5. \tag{15}$$

Let

$$w(t) = \frac{\rho(t)x^{(n-1)}(t)}{f\left(x\left[\frac{1}{2}\sigma(t)\right]\right)}.$$

Thus $w(t)$ satisfies

$$\begin{aligned}
\dot{w}(t) &\leqslant -\rho(t)q(t)\frac{f(x[g(t)])}{f(x[\sigma(t)/2])} + \frac{\dot{\rho}(t)}{\rho(t)}w(t) \\
&\quad -\tfrac{1}{2}f'\left(x\left[\tfrac{1}{2}\sigma(t)\right]\right)w(t)\frac{\dot{x}[\sigma(t)/2]\dot{\sigma}(t)}{f(x[\sigma(t)/2])} \\
&\leqslant -\rho(t)q(t) + \frac{\dot{\rho}(t)}{\rho(t)}w(t) - \tfrac{1}{2}M_{1/2}\alpha\frac{\sigma^{n-2}(t)\dot{\sigma}(t)}{\rho(t)}w^2(t) \\
&\leqslant -\rho(t)q(t) + \frac{\dot{\rho}^2(t)}{2M_{1/2}\alpha\sigma^{n-2}(t)\dot{\sigma}(t)\rho(t)} \\
&\quad -\left(\left(\tfrac{1}{2}M_{1/2}\alpha\frac{\sigma^{n-2}(t)\dot{\sigma}(t)}{\rho(t)}\right)^{1/2}w(t)\right. \\
&\qquad\qquad \left. - \frac{\dot{\rho}(t)/2/\rho(t)}{\left((M_{1/2}\alpha/2)\sigma^{n-2}(t)\dot{\sigma}(t)/\rho(t)\right)^{1/2}}\right)^2 \\
&\leqslant -\rho(t)q(t) + \frac{\dot{\rho}^2(t)}{2M_{1/2}\alpha\sigma^{n-2}(t)\dot{\sigma}(t)\rho(t)}.
\end{aligned}$$

Integrate the above inequality from t_5 to t to obtain

$$\int_{t_5}^{t}\left[\rho(s)q(s) - \frac{\dot{\rho}^2(s)}{2M_{1/2}\alpha\sigma^{n-2}(s)\dot{\sigma}(s)\rho(s)}\right]ds \leqslant w(t_5) - w(t) \leqslant w(t_5) < \infty$$

for all $t \leqslant t_5$. This contradicts (14). The case $x(t) < 0$ for $t \geqslant t_1 \geqslant t_0$ is similar and hence is omitted.

COROLLARY. *Let the condition* (14) *in Theorem* 2 *be replaced by*

$$\limsup_{t\to\infty} \int_{t_0}^{t}\rho(s)q(s)\,ds = \infty, \tag{16}$$

and

$$\lim_{t\to\infty}\int_{t_0}^{t}\frac{\dot\rho^2(s)}{\rho(s)\sigma^{n-2}(s)\dot\sigma(s)}\,ds<\infty. \tag{17}$$

Then the conclusion of Theorem 2 *holds.*

In the following theorem the function f is not required to be differentiable.

THEOREM 3. *Let the conditions* (3), (12) *and* (14) *in Theorem* 2 *be replaced by*

$$f\in C[R,R],\ xf(x)>0 \text{ and } f(x)/x\geqslant c_1>0 \text{ for } x\neq 0; \tag{18}$$

and

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[c_1\rho(s)q(s)-\frac{\dot\rho^2(s)}{2M_{1/2}\rho(s)\sigma^{n-2}(s)\dot\sigma(s)}\right]ds=\infty, \tag{19}$$

respectively, then the conclusion of Theorem 2 *holds.*

PROOF. Let $x(t)$ be a nonoscillatory solution of (1). Assume $x(t)>0$ for $t\geqslant t_1\geqslant t_0$. As in the proof of Theorem 2 we have

$$\dot x(t)>0 \quad\text{and}\quad x^{(n-1)}(t)>0 \quad \text{for } t\geqslant t_3,$$

and

$$\dot x[\sigma(t)/2]\geqslant M_{1/2}\sigma^{n-2}(t)x^{(n-1)}(t) \quad \text{for } t\geqslant t_5.$$

Letting

$$w(t)=\frac{\rho(t)x^{(n-1)}(t)}{x[\sigma(t)/2]},$$

we get

$$\dot w(t)\leqslant -c_1\rho(t)q(t)+\frac{\dot\rho(t)}{\rho(t)}w(t)-\tfrac{1}{2}M_{1/2}\frac{\sigma^{n-2}(t)\dot\sigma(t)}{\rho(t)}w^2(t).$$

The rest of the proof is similar to that of Theorem 2 and hence is omitted.

The following example is illustrative.

EXAMPLE 2. Consider the equations

$$x^{(n)}(t)+\gamma t^{-n}x[ct]=0 \quad \text{for } n \text{ even},\ t>0 \tag{20}$$

and

$$x^{(n)}(t)+\gamma t^{-n}x[ct]\exp(\sin x[ct])=0 \quad \text{for } n \text{ even},\ t>0, \tag{21}$$

where c and γ are positive constants. We let

$$\sigma(t)=\begin{cases} ct, & 0<c<1,\\ t, & c\geqslant 1.\end{cases}$$

Equation (20) is oscillatory by Theorem 2 for $\rho(t)=t^{n-1}$ and

$$\gamma>c^{1-n}2^{2n-3}(n-1)!\,(n-1)^2$$

and equation (21) is oscillatory by Theorem 3 for $\rho(t) = t^{n-1}$, $c_1 = e^{-1}$ and

$$\gamma > ec^{1-n}2^{2n-3}(n-1)!\,(n-1)^2.$$

It can easily be verified that Theorem 2.5(ii) in [2] and Theorem 1(ii) in [8] are not applicable to equation (20).

REMARKS. 1. Theorem 1 improves some of the results in [2, 4, 5, 7 and 8] for the case when $f(x) = x^{\alpha}\operatorname{sgn} x$, $0 < \alpha < 1$, while Theorems 2 and 3 unify and improve Theorem 1(ii) in [8] and Theorem 2.5(ii) in [2].

2. It is easy to check that our Theorems 2 and 3, when specialized to equation (11), turn out to be a substantial improvement on Theorem 3.2 in [9] and a result in [6].

3. It is clear that we do not stipulate that the function g in equation (1) be either retarded or advanced. Hence our theorems may hold for ordinary, retarded, advanced, and mixed type equations (see Examples 1 and 2 above).

4. The results of this paper can be extended to equations of the form

$$(22) \quad x^{(n)}(t) + \sum_{i=1}^{m} q_i(t)f_i(x[g_i(t)]) + F(t, x(t), x[g_1(t)], \ldots, x[g_m(t)]) = 0,$$

n is even, where $q_i, g_i \in C[[t_0, \infty), R]$, $f_i \in C[R, R]$, $q_i(t)$, $g_i(t)$ and $f_i(x)$, $1 \leqslant i \leqslant m$, satisfy the same assumptions we impose on $q(t)$, $g(t)$ and $f(x)$ respectively, and $F \in C[[t_0, \infty) \times R^{m+1}, R]$ satisfies

$$y_0F(t, y_0, y_1, \ldots, y_m) \geqslant 0 \quad \text{for } y_0y_i > 0,\ 1 \leqslant i \leqslant m.$$

ACKNOWLEDGEMENT. The authors are indebted to the referee for a number of helpful comments and suggestions.

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