

## OSCILLATION THEOREMS FOR $n$ TH ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

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**ABSTRACT.** In this note we study the oscillatory behavior of solutions of the  $n$ th order nonlinear functional differential equation

$$x^{(n)}(t) + q(t)f(x[g(t)]) = 0, \quad n \text{ even},$$

without assuming that the deviating argument is retarded or advanced. Sufficient conditions are established for all solutions of the equation to be oscillatory.

The purpose of this note is to obtain some oscillation criteria for the  $n$ th order equation

$$(1) \quad x^{(n)}(t) + q(t)f(x[g(t)]) = 0, \quad n \text{ even},$$

where the following conditions are assumed to hold:

- (2)  $q, g \in C[[t_0, \infty), R]$ ,  $q(t)$  is nonnegative and not identically zero on any interval of the form  $[t_1, \infty)$ ,  $t_1 \geq t_0$  and  $\lim_{t \rightarrow \infty} g(t) = \infty$ ;
- (3)  $f \in C[R, R]$ ,  $f$  is nondecreasing  $0 < f(x) \leq -f(-x)$  for  $x > 0$ ;
- (4)  $\int_0^\infty du/f(u) < \infty$  and  $\int_{-\infty}^0 du/f(u) < \infty$  for any  $\alpha > 0$ ;
- (5)  $f(xy) \geq Kf(x)f(y)$  for  $x, y > 0$  and  $K$  is a positive constant.

In what follows, by a proper solution of (1), we mean a function  $x \in C''[[T_x, \infty), R]$  which satisfies (1) for all sufficiently large  $t$  and  $\sup\{|x(t)| : t \geq T\} > 0$  for any  $T \geq T_x$ . A proper solution of (1) is called oscillatory if it has arbitrarily large zeros and it is called nonoscillatory otherwise.

We will have an occasion to use the following two lemmas given in [1].

**LEMMA 1.** *Let  $u$  be a positive and  $n$ -times differentiable function on an interval  $[t_0, \infty)$ . If  $u^{(n)}$  is of constant sign and not identically zero on any interval of the form  $[t_1, \infty)$ ,  $t_1 \geq t_0$ , then there exists a  $t_u \geq t_0$  and an integer  $l$ ,  $0 \leq l \leq n$  with  $n+l$  even for  $u^{(n)} \geq 0$  or  $n+l$  odd for  $u^{(n)} \leq 0$  and such that  $l > 0$  implies  $u^{(k)}(t) > 0$  for every  $t \geq t_u$  ( $k = 0, 1, \dots, l-1$ ) and  $l \leq n-1$  implies  $(-1)^{l+k} u^{(k)}(t) > 0$  for every  $t \geq t_u$  ( $k = l, l+1, \dots, n-1$ ).*

**LEMMA 2.** *If the function  $u$  is as in Lemma 1 and*

$$u^{(n-1)}(t)u^{(n)}(t) \leq 0 \quad \text{for every } t \leq t_u,$$

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then for every  $\lambda$ ,  $0 < \lambda < 1$ , there exists an  $M_\lambda > 0$  such that

$$u[\lambda t] \geq M_\lambda t^{n-1} |u^{(n-1)}(t)| \quad \text{for all large } t.$$

Moreover, if  $u$  is increasing, then

$$u(t) \geq u\left[\frac{1}{2}t\right] \geq M_{1/2} t^{n-1} |u^{(n-1)}(t)| \quad \text{for all large } t.$$

**REMARK.** From the proof of Lemma 2 [1, Lemma 2], we can take

$$M_{1/2} = \frac{2^{2-2^n}}{(n-1)!}.$$

We also need the following lemma.

**LEMMA 3.** Let  $n = 1$ , and conditions (2)–(4) hold. If

$$\int_A q(s) ds = \infty,$$

where  $A = \{t \in [t_0, \infty) : t_0 \leq g(t) < t\}$ , the retarded part of  $g(t)$ , then all proper solutions of (1) are oscillatory.

**PROOF.** The proof is similar to that of Theorem 2 in [3] and hence is omitted.

**THEOREM 1.** Let conditions (2)–(5) hold. Then all proper solutions of (1) are oscillatory if

$$(6) \quad \int_A q(s) f(g^{n-1}(s)) ds = \infty,$$

where  $A = \{t \in [t_0, \infty) : t_0 \leq g(t) < t\}$ , the retarded part of  $g(t)$ .

**PROOF.** Let  $x(t)$  be a nonoscillatory solution of (1), say  $x(t) > 0$  for  $t \geq t_1 \geq t_0$ . Then there exists  $t_2 \geq t_1$  such that  $x[g(t)] > 0$  for  $t \geq t_2$ .

By Lemma 1, there exists a  $t_3 \geq t_2$  such that

$$(7) \quad \dot{x}(t) > 0 \quad \text{and} \quad x^{(n-1)}(t) > 0 \quad \text{for } t \geq t_3.$$

Next, applying Lemma 2 for  $u = x$ , we derive that there exists a  $t_4 \geq t_3$  such that

$$x(t) \geq M_{1/2} t^{n-1} x^{(n-1)}(t) \quad \text{for } t \geq t_4.$$

Consequently, since  $g(t) \rightarrow \infty$  as  $t \rightarrow \infty$ , there exists a  $t_5 \geq t_4$  such that

$$(8) \quad x[g(t)] \geq M_{1/2} g^{n-1}(t) x^{(n-1)}[g(t)] \quad \text{for } t \geq t_5.$$

Thus (1) becomes

$$x^{(n)}(t) + q(t) f(M_{1/2} g^{n-1}(t) x^{(n-1)}[g(t)]) \leq 0 \quad \text{for } t \geq t_5.$$

Now, using (5) we get

$$x^{(n)}(t) + K^2 q(t) f(M_{1/2}) f(g^{n-1}(t)) f(x^{(n-1)}[g(t)]) \leq 0$$

or

$$(9) \quad x^{(n)}(t) + c q(t) f(g^{n-1}(t)) f(x^{(n-1)}[g(t)]) \leq 0$$

for some constant  $c > 0$ . Now if we let  $u(t) = x^{(n-1)}(t)$ , then (9) reduces to

$$\dot{u}(t) + cq(t)f(g^{n-1}(t))f(u[g(t)]) \leq 0.$$

Using Lemma 3, we obtain the desired contradiction.

If  $x(t) < 0$  for  $t \geq t_1 \geq t_0$ , then the transformation  $y = -x$  transforms (1) to

$$(1') \quad y^{(n)}(t) + q(t)f^*(y[g(t)]) = 0, \quad n \text{ even},$$

where  $f^*(y) = -f(-y)$ . Thus the above argument can be repeated on the positive solutions  $y(t)$  of (1'); this in turn yields the required results for negative solutions of (1).

For illustration consider the following example.

**EXAMPLE 1.** Consider the equation

$$(10) \quad x^{(6)}(t) + (t-1)^{-8/3}x^{1/3}[t + \sin t] = 0, \quad t \geq \pi.$$

The retarded part of  $g(t) = t + \sin t$  is  $A = \bigcup_{k=1}^{\infty} ((2k-1)\pi, 2k\pi)$ .

$$\begin{aligned} \int_A q(s)f(g^{n-1}(s)) ds &= \sum_{k=1}^{\infty} \int_{(2k-1)\pi}^{2k\pi} \frac{(t + \sin t)^{5/3}}{(t-1)^{8/3}} dt \\ &\geq \sum_{k=1}^{\infty} \int_{(2k-1)\pi}^{2k\pi} \frac{(t-1)^{5/3}}{(t-1)^{8/3}} dt \\ &= \sum_{k=1}^{\infty} \log \frac{2k\pi - 1}{2k\pi - \pi - 1} = \infty, \end{aligned}$$

and so from Theorem 1, it follows that all proper solutions of (1) are oscillatory.

The following theorem extends and improves Theorem 3.2 of Wong [9] who considered the equation

$$(11) \quad \ddot{x}(t) + q(t)x[g(t)] = 0,$$

where  $ct \leq g(t) \leq t$ ,  $c > 0$  and  $q(t) \geq 0$ .

**THEOREM 2.** *Let conditions (2) and (3) hold, and*

$$(12) \quad f'(x) > \alpha > 0 \quad \text{for } x \neq 0 \quad (' = d/dt).$$

*Suppose further that there exist functions  $\rho, \sigma \in C^1[[t_0, \infty), (0, \infty)]$  such that*

$$(13) \quad \sigma(t) \leq \min\{t, 2g(t)\}, \dot{\sigma}(t) > 0 \text{ and } \sigma(t) \rightarrow \infty \text{ as } t \rightarrow \infty \quad (' = d/dt);$$

*and*

$$(14) \quad \lim_{t \rightarrow \infty} \sup \int_{t_0}^t \left[ \rho(s)q(s) - \frac{\dot{\rho}^2(s)}{2\alpha M_{1/2}\sigma^{n-2}(s)\dot{\sigma}(s)\rho(s)} \right] ds = \infty,$$

*where  $M_{1/2} = 2^{2-2n}/(n-1)!$  ( $M_{1/2}$  is as in Lemma 2 for  $\lambda = \frac{1}{2}$ ). Then every solution of equation (1) is oscillatory.*

**PROOF.** Let  $x(t)$  be a nonoscillatory solution of (1), say  $x(t) > 0$  for  $t \geq t_1 \geq t_0$ . It follows, as in the proof of Theorem 1 that there exists a  $t_3 \geq t_2$  so that

$$\dot{x}(t) > 0 \quad \text{and} \quad x^{(n-1)}(t) > 0 \quad \text{for } t \geq t_3.$$

Now, applying Lemma 2 for  $u = \dot{x}$  and  $\lambda = \frac{1}{2}$ , we easily derive that there exists  $t_4 \geq t_3$  such that

$$\dot{x}\left[\frac{1}{2}t\right] \geq M_{1/2}t^{n-2}x^{(n-1)}(t) \quad \text{for } t \geq t_4.$$

Since  $\sigma(t) \rightarrow \infty$  as  $t \rightarrow \infty$ , there exists  $t_5 \geq t_4$  such that

$$\dot{x}\left[\frac{1}{2}\sigma(t)\right] \geq M_{1/2}\sigma^{n-2}(t)x^{(n-1)}[\sigma(t)] \quad \text{for } t \geq t_5.$$

Now, by condition (13) and the fact that  $x^{(n-1)}(t)$  is nonincreasing, we have

$$(15) \quad \dot{x}\left[\frac{1}{2}\sigma(t)\right] \geq M_{1/2}\sigma^{n-2}(t)x^{(n-1)}(t) \quad \text{for } t \geq t_5.$$

Let

$$w(t) = \frac{\rho(t)x^{(n-1)}(t)}{f(x[\frac{1}{2}\sigma(t)])}.$$

Thus  $w(t)$  satisfies

$$\begin{aligned} \dot{w}(t) &\leq -\rho(t)q(t)\frac{f(x[g(t)])}{f(x[\sigma(t)/2])} + \frac{\dot{\rho}(t)}{\rho(t)}w(t) \\ &\quad - \frac{1}{2}f'(x[\frac{1}{2}\sigma(t)])w(t)\frac{\dot{x}[\sigma(t)/2]\dot{\sigma}(t)}{f(x[\sigma(t)/2])} \\ &\leq -\rho(t)q(t) + \frac{\dot{\rho}(t)}{\rho(t)}w(t) - \frac{1}{2}M_{1/2}\alpha\frac{\sigma^{n-2}(t)\dot{\sigma}(t)}{\rho(t)}w^2(t) \\ &\leq -\rho(t)q(t) + \frac{\dot{\rho}^2(t)}{2M_{1/2}\alpha\sigma^{n-2}(t)\dot{\sigma}(t)\rho(t)} \\ &\quad - \left( \left( \frac{1}{2}M_{1/2}\alpha\frac{\sigma^{n-2}(t)\dot{\sigma}(t)}{\rho(t)} \right)^{1/2} w(t) \right. \\ &\quad \left. - \frac{\dot{\rho}(t)/2/\rho(t)}{\left( (M_{1/2}\alpha/2)\sigma^{n-2}(t)\dot{\sigma}(t)/\rho(t) \right)^{1/2}} \right)^2 \\ &\leq -\rho(t)q(t) + \frac{\dot{\rho}^2(t)}{2M_{1/2}\alpha\sigma^{n-2}(t)\dot{\sigma}(t)\rho(t)}. \end{aligned}$$

Integrate the above inequality from  $t_5$  to  $t$  to obtain

$$\int_{t_5}^t \left[ \rho(s)q(s) - \frac{\dot{\rho}^2(s)}{2M_{1/2}\alpha\sigma^{n-2}(s)\dot{\sigma}(s)\rho(s)} \right] ds \leq w(t_5) - w(t) \leq w(t_5) < \infty$$

for all  $t \leq t_5$ . This contradicts (14). The case  $x(t) < 0$  for  $t \geq t_1 \geq t_0$  is similar and hence is omitted.

**COROLLARY.** Let the condition (14) in Theorem 2 be replaced by

$$(16) \quad \lim_{t \rightarrow \infty} \sup \int_{t_0}^t \rho(s)q(s) ds = \infty,$$

and

$$(17) \quad \lim_{t \rightarrow \infty} \int_{t_0}^t \frac{\dot{\rho}^2(s)}{\rho(s)\sigma^{n-2}(s)\dot{\sigma}(s)} ds < \infty.$$

Then the conclusion of Theorem 2 holds.

In the following theorem the function  $f$  is not required to be differentiable.

**THEOREM 3.** Let the conditions (3), (12) and (14) in Theorem 2 be replaced by

$$(18) \quad f \in C[R, R], xf(x) > 0 \text{ and } f(x)/x \geq c_1 > 0 \text{ for } x \neq 0;$$

and

$$(19) \quad \lim_{t \rightarrow \infty} \sup \int_{t_0}^t \left[ c_1 \rho(s) q(s) - \frac{\dot{\rho}^2(s)}{2M_{1/2}\rho(s)\sigma^{n-2}(s)\dot{\sigma}(s)} \right] ds = \infty,$$

respectively, then the conclusion of Theorem 2 holds.

**PROOF.** Let  $x(t)$  be a nonoscillatory solution of (1). Assume  $x(t) > 0$  for  $t \geq t_1 \geq t_0$ . As in the proof of Theorem 2 we have

$$\dot{x}(t) > 0 \quad \text{and} \quad x^{(n-1)}(t) > 0 \quad \text{for } t \geq t_3,$$

and

$$\dot{x}[\sigma(t)/2] \geq M_{1/2}\sigma^{n-2}(t)x^{(n-1)}(t) \quad \text{for } t \geq t_5.$$

Letting

$$w(t) = \frac{\rho(t)x^{(n-1)}(t)}{x[\sigma(t)/2]},$$

we get

$$\dot{w}(t) \leq -c_1\rho(t)q(t) + \frac{\dot{\rho}(t)}{\rho(t)}w(t) - \frac{1}{2}M_{1/2}\frac{\sigma^{n-2}(t)\dot{\sigma}(t)}{\rho(t)}w^2(t).$$

The rest of the proof is similar to that of Theorem 2 and hence is omitted.

The following example is illustrative.

**EXAMPLE 2.** Consider the equations

$$(20) \quad x^{(n)}(t) + \gamma t^{-n}x[ct] = 0 \quad \text{for } n \text{ even, } t > 0$$

and

$$(21) \quad x^{(n)}(t) + \gamma t^{-n}x[ct] \exp(\sin x[ct]) = 0 \quad \text{for } n \text{ even, } t > 0,$$

where  $c$  and  $\gamma$  are positive constants. We let

$$\sigma(t) = \begin{cases} ct, & 0 < c < 1, \\ t, & c \geq 1. \end{cases}$$

Equation (20) is oscillatory by Theorem 2 for  $\rho(t) = t^{n-1}$  and

$$\gamma > c^{1-n}2^{2n-3}(n-1)!(n-1)^2$$

and equation (21) is oscillatory by Theorem 3 for  $\rho(t) = t^{n-1}$ ,  $c_1 = e^{-1}$  and

$$\gamma > ec^{1-n}2^{2n-3}(n-1)!(n-1)^2.$$

It can easily be verified that Theorem 2.5(ii) in [2] and Theorem 1(ii) in [8] are not applicable to equation (20).

**REMARKS.** 1. Theorem 1 improves some of the results in [2, 4, 5, 7 and 8] for the case when  $f(x) = x^\alpha \operatorname{sgn} x$ ,  $0 < \alpha < 1$ , while Theorems 2 and 3 unify and improve Theorem 1(ii) in [8] and Theorem 2.5(ii) in [2].

2. It is easy to check that our Theorems 2 and 3, when specialized to equation (11), turn out to be a substantial improvement on Theorem 3.2 in [9] and a result in [6].

3. It is clear that we do not stipulate that the function  $g$  in equation (1) be either retarded or advanced. Hence our theorems may hold for ordinary, retarded, advanced, and mixed type equations (see Examples 1 and 2 above).

4. The results of this paper can be extended to equations of the form

$$(22) \quad x^{(n)}(t) + \sum_{i=1}^m q_i(t)f_i(x[g_i(t)]) + F(t, x(t), x[g_1(t)], \dots, x[g_m(t)]) = 0,$$

$n$  is even, where  $q_i, g_i \in C[[t_0, \infty), R]$ ,  $f_i \in C[R, R]$ ,  $q_i(t)$ ,  $g_i(t)$  and  $f_i(x)$ ,  $1 \leq i \leq m$ , satisfy the same assumptions we impose on  $q(t)$ ,  $g(t)$  and  $f(x)$  respectively, and  $F \in C[[t_0, \infty) \times R^{m+1}, R]$  satisfies

$$y_0 F(t, y_0, y_1, \dots, y_m) \geq 0 \quad \text{for } y_0 y_i > 0, 1 \leq i \leq m.$$

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