WEIGHTED GROUP ALGEBRA AS AN IDEAL
IN ITS SECOND DUAL SPACE

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ABSTRACT. For a locally compact group G let $L^1(G, \omega\lambda)$ be a weighted group algebra. We characterize compact and weakly compact multipliers on $L^1(G, \omega\lambda)$. This characterization is employed to find a necessary and sufficient condition for $L^1(G, \omega\lambda)$ to be an ideal in its second dual space, where the second dual is equipped with an Arens product. In the special case where $\omega(t) = 1$ ($t \in G$), we deduce a result due to K. P. Wong that if G is a compact group, then $L^1(G, \lambda)$ is an ideal in its second dual space and its converse due to S. Watanabe.

1. Introduction. Following the work of W. G. Bade and H. G. Dales [4] on compact elements of $L^1(R^+, \omega)$, in [9] we have characterized elements $g \in L^1(G, \omega\lambda)$ for which the operator ρ_g: $f \mapsto f * g$ ($f \in L^1(G, \omega\lambda)$) is compact, where we have called them compact elements. In this paper we prove that every right (resp. left) compact multiplier T on $L^1(G, \omega\lambda)$ is represented by a right (resp. left) compact element $g \in L^1(G, \omega\lambda)$, such that $T(f) = f * g$ ($f \in L^1(G, \omega\lambda)$), and for the left multiplier T we have $T(f) = g * f$ ($f \in L^1(G, \omega\lambda)$). We also prove that every right (resp. left) weakly compact multiplier is a right (resp. left) compact multiplier.

K. P. Wong in [17] has proved that if G is a compact group and if the second dual space of $L^1(G, \lambda)$ is equipped with an Arens product, then the canonical image of $L^1(G, \lambda)$ in its second dual space is an ideal. For the converse of the Wong's result S. Watanabe gave two different proofs [15 and 16]. Other proofs were also provided by M. Grosser [11] and D. L. Johnson [12].

Using our characterization of the weakly compact elements, we find a necessary and sufficient condition for a weighted group algebra to be an ideal in its second dual space, where the second dual is equipped with an Arens product. In particular, we obtain other proofs for the results of Wong and Watanabe.

2. Preliminaries. Throughout, G will be a locally compact group and λ a fixed left Haar measure on G. A positive and continuous function ω on G satisfying $\omega(st) \leq \omega(s)\omega(t)$ ($s, t \in G$) will be called a weight function. For a weight function ω, let $L^1(G, \omega\lambda)$ be the Banach space of all complex functions f such that

$$\|f\| = \int_G |f(t)|\omega(t) \, d\lambda(t) < \infty,$$

where as usual we equate functions λ almost everywhere. Under convolution product

$$(f * g)(x) = \int_G f(xy^{-1}) g(y) \, d\lambda(y) \qquad (f, g \in L^1(G, \omega\lambda), \lambda \text{ a.e. } x \in G),$$

$L^1(G, \omega\lambda)$ becomes a Banach algebra. When $\omega(t) \geq 1$, the algebra $L^1(G, \omega\lambda)$ is called a Beurling algebra [13].

As in [8 and 9] we appeal to a larger algebra of measures; let $M(G, \omega)$ be the Banach space of all complex regular Borel measures μ on G such that

$$\|\mu\| = \int_G \omega(t) \, d|\mu|(t) < \infty.$$

If $C_0(G, \omega)$ is the Banach space of all functions f on G such that $f/\omega \in C_0(G)$ and $\|f\| = \sup_{x \in G} |f(x)/\omega(x)|$, then by the pairing

$$\langle \mu, \psi \rangle = \int_G \psi(x) \, d\mu(x) \qquad (\mu \in M(G, \omega), \psi \in C_0(G, \omega)),$$

we have $(C_0(G, \omega))' = M(G, \omega)$, and we can define the product of $\mu, \nu \in M(G, \omega)$ by

$$\int_G \psi(x) \, d(\mu * \nu)(x) = \int_G \int_G \psi(xy) \, d\mu(x) \, d\nu(y) \qquad (\psi \in C_0(G, \omega)).$$

Let X be a Banach space and X' be the dual of X. If $w = \sigma(X, X')$ is the weak topology on X, and U is the open unit ball of X, we call an operator T on X weakly compact if the weak closure of $T(U)$ in X is weakly compact. On $M(G, \omega)$ we consider two topologies other than the norm topology and the weak topology: they are (I) the topology (so) in which a net (μ_α) tends to a measure μ (so) if $\mu_\alpha * f \overset{\|\cdot\|}{\to} \mu * f$, for every $f \in L^1(G, \omega\lambda)$, and (II) the topology

$$w^* = \sigma(M(G, \omega), C_0(G, \omega)).$$

The map $f \mapsto \mu_f$, where $d\mu_f(x) = f(x) \, d\lambda(x)$, defines an isometric isomorphism from $L^1(G, \omega\lambda)$ into $M(G, \omega)$, and $L^1(G, \omega\lambda)$ can be identified with a closed ideal of $M(G, \omega)$.

We call an element $g \in L^1(G, \omega\lambda)$ right compact (resp. right weakly compact) if the operator $\rho_g \colon f \mapsto f * g$ is compact (resp. weakly compact). We define left (resp. left weakly) compact elements by replacing $f * g$ by $g * f$ in the above definition.

LEMMA 2.1. *The algebra* $L^1(G, \omega\lambda)$ *possesses a bounded approximate identity* (f_i) *such that each* f_i *is a continuous function with compact support and there exists a compact set* S *containing the supports of all the* f_i.

PROOF. It is well known that the algebra $L^1(G, \lambda)$ has a bounded approximate identity (f_i) consisting of continuous functions with compact support, where for each f_i the support of f_i contains the identity element e of G [5, Proposition 69.16]. Moreover, we can assume that there exists a compact set S containing the supports of all the f_i. The net (f_i) can then, for every weight function ω, serve as a bounded approximate identity for $L^1(G, \omega\lambda)$. Indeed, if $g \in L^1(G, \omega\lambda)$ is with compact

support K, then since $e \in S$, we have $K \subset KS$, whence,

$$\|g - g * f_i\|_\omega = \int_G |g(t) - (g * f_i)(t)| \omega(t) \, d\lambda(t)$$

$$\leqslant \left(\sup_{x \in KS} \omega(x) \right) \int_G |g(t) - (g * f_i)(t)| \, d\lambda(t)$$

$$= \left(\sup_{x \in KS} \omega(x) \right) \|g - g * f_i\|_1.$$

Likewise,

$$\|g - f_i * g\|_\omega \leqslant \left(\sup_{x \in SK} \omega(x) \right) \|g - f_i * g\|_1.$$

Since the set of functions with compact support is dense in $L^1(G, \omega\lambda)$, the lemma is proved.

LEMMA 2.2. *Suppose* (f_i) *is a bounded approximate identity for* $L^1(g, \omega\lambda)$ *as described in Lemma* 2.1. *Then* $f_i \to \delta_e$ *in the* w^*-*topology of* $M(G, \omega)$, *where* e *is the identity element of* G *and* δ_e *is the unit mass concentrated at* e.

PROOF. First we consider the case $\omega(t) = 1$ $(t \in G)$. We have $f_i * g \to g = \delta_e * g$, in the norm topology of $L^1(G, \lambda)$. Hence $f_i \to \delta_e$ in the w^*-topology of $M(G)$ [**10**, Lemma 1.1.1]. Thus, in this case we have

$$(1) \qquad \int_G \psi(x) f_i(x) \, d\lambda(x) \to \psi(e) \qquad (\psi \in C_0(G)).$$

For a general ω, let $\varphi \in C_0(G, \omega)$. Then

$$(2) \qquad \langle \varphi, f_i \rangle = \int_G \varphi(x) f_i(x) \, d\lambda(x).$$

Since the f_i vanish off the compact set S, the function φ can be assumed to vanish at infinity. Then, from (1) and (2) it follows $\langle \varphi, f_i \rangle \to \varphi(e) = \langle \varphi, \delta_e \rangle$, and the lemma is proved.

LEMMA 2.3. *Suppose* T *is a right* (*resp. left*) *multiplier on* $L^1(G, \omega\lambda)$. *Then there exists* $\mu \in M(G, \omega)$ *such that*

$$T(f) = f * \mu \qquad (f \in L^1(G, \omega\lambda)),$$

(*resp.* $T(f) = \mu * f$ $(f \in L^1(G, \omega\lambda))$).

PROOF. We consider the right multiplier case. Let $\{f_i : i \in I\}$ be a bounded approximate identity for $L^1(G, \omega\lambda)$. The set $\{T(f_i) : i \in I\}$ is a bounded subset of $M(G, \omega)$. Thus, by the Banach-Alaoglu theorem, there exists a subnet $\{f_j : j \in J\}$ of $\{f_i : i \in I\}$ and a measure $\mu \in M(G, \omega)$, such that $T(f_j) \to \mu$ in the ω^*-topology of $M(G, \omega)$. Now if $f \in L^1(G, \omega\lambda)$, then

$$T(f * f_j) \overset{\|\cdot\|}{\to} T(f) \quad \text{and} \quad T(f * f_j) = f * T(f_j) \overset{w^*}{\to} f * \mu,$$

and the lemma is proved.

3. Compact and weakly compact multipliers.

THEOREM 3.1. *The following are equivalent.*

(I) *T is a weakly compact right multiplier on* $L^1(G, \omega\lambda)$.

(II) *T is a compact right multiplier on* $L^1(G, \omega\lambda)$.

(III) *There exists* $g \in L^1(G, \omega\lambda)$ *such that* $T(f) = f * g$ ($f \in L^1(G, \omega\lambda)$) *and such that the function*

$$x \mapsto \int_G \frac{\omega(xy)}{\omega(x)} |g(y)| d\lambda(y) \qquad (x \in G)$$

vanishes at infinity.

PROOF. (I) $\Rightarrow$ (II). By Lemma 2.3 there exists $\mu \in M(G, \omega)$ such that $T(f) = f * \mu$ ($f \in L^1(G, \omega\lambda)$). First, we show that $\mu \in L^1(G, \omega\lambda)$. Let $\{f_i : i \in I\}$ be a bounded approximate identity for $L^1(G, \omega\lambda)$, same as in Lemma 2.1. By weak compactness of T there exist a subnet (f_j) of (f_i) and $g \in L^1(G, \omega\lambda)$ such that $T(f_j) = f_j * \mu \to g$ in the weak topology of $L^1(G, \omega\lambda)$. By Lemma 2.2, $f_j * \mu \overset{w^*}{\to} \delta_0 * \mu = \mu$. Hence $\mu = g \in L^1(G, \omega\lambda)$.

Next, writing g instead of μ, we first prove that for every $i \in I$, the operator $h \mapsto h * g * f_i$ ($h \in L^1(G, \omega\lambda)$) is a compact operator [**4**, Theorem 2.9].

Let (k_n) be a bounded sequence in $L^1(G, \omega\lambda)$. Since $h \mapsto h * g$ is weakly compact, there is a subsequence (k_{n_i}) of (k_n) and an element k_0 of $L^1(G, \omega\lambda)$ with $k_{n_i} * g \to k_0$ weakly. Hence, to show that $h \mapsto h * g * f_i$ is compact, it suffices to show that, if $h_n \to h_0$ weakly in $L^1(G, \omega\lambda)$, then $h_n * f_i \to h_0 * f_i$ in the norm topology of $L^1(G, \omega\lambda)$. Suppose then that $h_n \to h_0$ weakly. For each $x \in G$,

$$\left[f_i * (h_n - h_0) \right](x) = \int_G f_i(xy^{-1}) \left[h_n(y) - h_0(y) \right] d\lambda(y).$$

If $\varphi(y) = f_i(xy^{-1})$ ($y \in G$), then φ is bounded and has compact support, and so $\sup\{|\varphi(y)/\omega(y)| : y \in G\} < \infty$. Thus, $\varphi \in L^\infty(G, \omega\lambda)$, and so $(h_n * f_i)(x) \to (h_0 * f_i)(x)$, as $n \to \infty$. By [**7**, III, 6.13], $h_n * f_i \to h_0 * f_i$, as $n \to \infty$, in measure on subsets of G of finite measure (for the measure $\omega(t) d\lambda(t)$). But $h_n * f_i \to h_0 * f_i$ weakly as $n \to \infty$, and so by [**7**, IV, 8.12], $h_n * f_i \to h_0 * f_i$ in $L^1(G, \omega\lambda)$, as $n \to \infty$. Now, we have $g = \lim g * f_i$ in $L^1(G, \omega\lambda)$, and the compactness of T follows.

Obviously (II) $\Rightarrow$ (I). Thus, (I) and (II) are equivalent.

To prove the equivalence of (II) and (III), we note that if T is a compact multiplier on $L^1(G, \omega\lambda)$, then by an argument similar to the one in the first paragraph there is a $g \in L^1(G, \omega\lambda)$ such that $T(f) = f * g$ ($f \in L^1(G, \omega\lambda)$). Then, the equivalence of (II) and (III) follows from [**9**, Theorem 1].

A similar result holds for left multipliers.

COROLLARY 3.2 (SAKAI [**14**, THEOREM 1]). *If G is a locally compact noncompact group, then* 0 *is the only right* (*resp. left*) *weakly compact element of* $L^1(G, \lambda)$.

PROOF. This follows from Theorem 3.1 and [**9**, Theorem 1].

4. $L^1(G, \omega\lambda)$ as an ideal. Suppose A is a Banach algebra, then the second dual space A'' of A may be equipped with two Banach algebra products, known as first (or left) and second (or right) Arens products [2, 3 and 6] each of which is an extension of the original product in A as canonically embedded in A''. We assume A'' is equipped with the left Arens product. For each $a \in A$, we let $\pi(a)$ denote the canonical image of a in A'', and ρ_a and λ_a be the operators $\rho_a: b \mapsto ba$ ($b \in A$) and $\lambda_a: b \mapsto ab$ ($b \in A$).

Below we give a different proof to a result due to Watanabe [15].

PROPOSITION 4.1. *Let A be a Banach algebra. Then the following statements are equivalent.*

(I) *For each $a \in A$, ρ_a (resp. λ_a) is right (resp. left) weakly compact on A.*

(II) *$\pi(A)$ is a right (resp. left) ideal of A''.*

PROOF. For every $a \in A$ the operator ρ_a is weakly compact if and only if $(\rho_a)^{**}A'' \subset \pi(A)$, where ρ_a^{**} is the second adjoint of ρ_a [7, VI, 4.2]. But ρ_a^{**} is the same as multiplication on the right by $\pi(a)$ in A''. Hence for every $a \in A$, ρ_a is weakly compact if and only if $A''\pi(a) \subset \pi(A)$. The corresponding result for λ_a can, similarly, be proved and the equivalence of (I) and (II) is established.

COROLLARY 4.2. *A necessary and sufficient condition for the canonical image of $L^1(G, \omega\lambda)$ in its second dual to be a right (resp. left) ideal is that for every $g \in L^1(G, \omega\lambda)$, the function*

$$x \mapsto \int_G \frac{\omega(xy)}{\omega(x)} |g(y)| d\lambda(y) \qquad (x \in G),$$

(resp. $x \mapsto \int_G \omega(yx)/\omega(x)|g(y)| \, d\lambda(y)$ ($x \in G$)) vanishes at infinity.

PROOF. This follows from Theorem 3.2 and Proposition 4.1.

COROLLARY 4.3. *If G is a compact group, then the canonical image of $L^1(G, \omega\lambda)$ in its second dual is an ideal.*

PROOF. For every $g \in L^1(G, \omega\lambda)$, ρ_g and λ_g are compact [9, Corollary 1], and hence weakly compact.

If in the above corollary we let $\omega(t) = 1$ ($t \in G$), then we obtain Wong's result.

COROLLARY 4.4 (S. WATANABE [15, 16]). *If $L^1(G, \lambda)$ is a right (resp. left) ideal in its second dual space, then G is compact.*

PROOF. If $L^1(G, \lambda)$ is a right (resp. left) ideal in its second dual space, then by Proposition 4.1, for every $g \in L^1(G, \lambda)$, ρ_g (resp. λ_g) is weakly compact, and the result follows from Corollary 3.2.

REMARK 4.5. We have seen that for the case $\omega(t) = 1$ ($t \in G$), if $0 \neq g \in L^1(G, \lambda)$, then ρ_g is compact if and only if λ_g is compact (in which case G is a compact group). On the other hand, if for some $0 \neq g \in L^1(G, \lambda)$, ρ_g is compact, then this holds for every element of $L^1(G, \lambda)$. These considerations amount to the following question.

Question 1. Is every right (resp. left) compact element of $L^1(G, \omega\lambda)$ a left (resp. right) compact element?

Question 2. Does from right (resp. left) compactness of one nonzero element follow that all the other elements of $L^1(G, \omega\lambda)$ are right (resp. left) compact elements?

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