PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 90, Number 1, January 1984

TARSKI'S EXTENSION THEOREM FOR GROUP-VALUED CHARGES

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ABSTRACT. A result of Tarski on extensions of real-valued charges is extended to group-valued charges for certain groups.

1. Introduction and notation. A finitely additive function on a field of sets taking values in a group will be called a charge. The purpose of this paper is to examine and extend the validity of the following theorem of Tarski (see [**3** and **1**]) for group-valued charges. 2^X is the power set of a set X.

TARSKI'S THEOREM. *If $\mathcal{C}$ is a field of subsets of a set X, then any real-valued charge on $\mathcal{C}$ can be extended as a real-valued charge on 2^X.*

We assume all groups are commutative, though this is unnecessary. For any collection $\mathcal{F}$ of subsets of X, $\langle \mathcal{F} \rangle$ is the field generated by $\mathcal{F}$. $|A|$ is the cardinality of a set A.

2. Results. We start with an elementary lemma.

LEMMA 1. *Let $\mathcal{C}$ be a finite field of subsets of X, G a group, μ a G-valued charge on $\mathcal{C}$, and $A \notin \mathcal{C}$. Then there is a G-valued charge on $\langle \mathcal{C}, A \rangle$ which is an extension of μ.*

PROOF. We shall exhibit a charge τ on $\mathcal{P}(X)$ which extends μ. Since $\mathcal{C}$ is a finite field there is a partition $A_1, A_2, \ldots, A_k$ of X consisting of nonempty sets from $\mathcal{C}$ such that $\mathcal{C}$ is the collection of all possible unions of sets from this partition. Fix points $x_i \in A_i$ for $i = 1, \ldots, n$. For any $A \subset X$, if we define $\tau(A) = \Sigma_{x_i \in A} \mu(A_i)$, then τ is a G-valued charge on $\mathcal{P}(X)$ which extends μ.

THEOREM 2. *Let G be a compact T_2 topological group. Let μ be a G-valued charge defined on a field $\mathcal{C}$ of subsets of a set X. Then μ can be extended to a G-valued charge on 2^X.*

PROOF. Let $A \notin \mathcal{C}$. We shall show that μ can be extended as a G-valued charge to $\langle \mathcal{C}, A \rangle$. Then the extension to 2^X follows by the usual transfinite argument. Let $\mathcal{B} = \langle \mathcal{C}, A \rangle$.

Received by the editors July 8, 1982 and, in revised form, April 25, 1983.
1980 *Mathematics Subject Classification*. Primary 28B10.

[1]This work was done during the second author's visit to the University of Naples (September–December 1981) supported by C.N.R.

Let $G^{\mathcal{B}}$ be the set of all functions defined on $\mathcal{B}$ taking values in G and equipped with the product topology. Then $G^{\mathcal{B}}$ is a compact T_2 space. For any field $\mathcal{D} \subset \mathcal{B}$, let $L_{\mathcal{D}}$ be the set of all elements of $G^{\mathcal{B}}$ which are additive on $\mathcal{D}$, i.e.,

$$L_{\mathcal{D}} = \bigcap_{\substack{C, D \in \mathcal{D} \\ C \cap D = \varnothing}} \left\{ \tau \in G^{\mathcal{B}} : \tau(C \cup D) = \tau(C) + \tau(D) \right\}.$$

Then clearly $L_{\mathcal{D}}$ is a closed subset of $G^{\mathcal{B}}$.

For any finite field $\mathcal{F} \subset \mathcal{C}$ if we define

$$M_{\mathcal{F}} = \left\{ \tau \in L_{\langle \mathcal{F}, A \rangle} : \tau(B) = \mu(B) \text{ for all } B \in \mathcal{F} \right\}$$

then $M_{\mathcal{F}}$ is a closed subset of $G^{\mathcal{B}}$ and is nonempty by Lemma 1. Also the family $\{M_{\mathcal{F}}: \mathcal{F}$ is a finite subfield of $\mathcal{C}\}$ has the finite intersection property. Since $G^{\mathcal{B}}$ is compact, there is a $\tau \in M_{\mathcal{F}}\ \forall \mathcal{F} \subset \mathcal{C}$ which is a G-valued charge on $\mathcal{B}$ extending μ because $\bigcup \langle \mathcal{F}, A \rangle$, the union over all $\mathcal{F}$, equals $\mathcal{B}$.

REMARK 3. Theorem 2 is also valid for any algebraically compact group (see **[2]** for the definition and properties). If G is algebraically compact there is a compact T_2 topological group H containing G and K, a subgroup of H, such that $G \oplus K = H$. Considering μ as an H-valued charge, by the previous case, we extend μ to an H-valued charge τ on 2^X. Now, if we compose τ with the projection to G from H, we get a G-valued charge which is an extension of μ. Since every divisible group is algebraically compact, Theorem 2 is also valid for any divisible group, a fact which can also be proved otherwise.

REMARK 4. If μ is a charge on a field of sets $\mathcal{B}$ taking values in a compact T_2 topological group, then μ need not be exhaustive (i.e. $\mu(A_n)$ need not converge to 0 for every pairwise disjoint sequence of sets A_n, $n \geqslant 1$, from $\mathcal{B}$). Let $G = \{0, 1\}^{\aleph_0}$ be the countable product of the discrete two-point topological group $\{0, 1\}$. Let $g_1, g_2, \ldots$ be a countable dense subset of G. Let $\mathcal{B}$ be the finite cofinite field on the set of positive integers. Define μ on $\mathcal{B}$ by

$$\begin{aligned} \mu(B) &= \sum_{i \in B} g_i \quad \text{if } B \text{ is finite,} \\ &= \sum_{i \notin B} g_i \quad \text{if } B \text{ is cofinite.} \end{aligned}$$

Then μ is a charge on $\mathcal{B}$ taking values in a compact T_2 group which is not exhaustive. Thus Theorem 2 does not follow from the many extension theorems which followed **[5]** (see for example Theorem 3 of **[4]**).

REMARK 5. Since any finite group is a compact T_2 group, Theorem 2 holds for any finite group. If G is the two-element group $\{0, 1\}$, or more generally, if every element of G is of order 2, we can prove Theorem 2 without resorting to topological arguments. This is so because a function μ on a field $\mathcal{C}$ into G is a charge if and only if $\mu(A \triangle B) = \mu(A) + \mu(B)\ \forall A, B \in \mathcal{C}$. Hence, a function μ on a field $\mathcal{C}$ into G is a charge if and only if it is a group homomorphism from $\mathcal{C}$ considered as a group with $\triangle$.

As a final result, we shall generalize Lemma 1 by replacing the finiteness of $\mathcal{C}$ with the finiteness of the range of μ.

THEOREM 6. *Let G be a group. Let μ be a G-valued charge on a field $\mathcal{C}$ of subsets of X whose range is a finite set. Then μ can be extended as a G-valued charge to any field $\mathcal{B}$ containing $\mathcal{C}$.*

PROOF. Let $\mu(\mathcal{C})$ be the range of μ on $\mathcal{C}$. Let H be the subgroup of all elements of G of finite order.

Case (i). $\mu(\mathcal{C}) \subset H$. In this case the subgroup K generated by $\mu(\mathcal{C})$ is finite and μ can be considered as a K-valued charge on $\mathcal{C}$, and by Theorem 3 there is a K-valued charge on $\mathcal{B}$ extending μ.

Case (ii). $\mu(\mathcal{C}) \not\subset H$, i.e., $\mu(\mathcal{C}) - H \neq \varnothing$. Let

$$\mathcal{P} = \big\{\{A_1,\ldots,A_n\} : A_i \neq \varnothing, A_i \in \mathcal{C}, \mu(A_i) \notin H \\ \text{for } i = 1,\ldots,n \text{ and } A_i \cap A_j = \varnothing \text{ for } i \neq j\big\}.$$

Then if $\{A_1,\ldots,A_n\} \in \mathcal{P}$, for any fixed $a \notin H$, $|\{i\colon \mu(A_i) = a\}| \leqslant |\mu(\mathcal{C})|$, so $n \leqslant |\mu(\mathcal{C})|^2$. Let $\{A_1,\ldots,A_p\} \in \mathcal{P}$ be such that p is the maximum possible n for collections $\{A_1,\ldots,A_n\}$ in $\mathcal{P}$. We shall show that $\mu|_{A_i \cap \mathcal{C}}$ can be extended to $A_i \cap \mathcal{B}$ as a G-valued charge.

Let us concentrate on A_1. Consider

$$\mathcal{F} = \{B \subset A_1 : B \in \mathcal{B} \text{ and } \mu(B) \notin H\}.$$

Then (i) $A_1 \in \mathcal{F}$; (ii) $B \in \mathcal{F}$, $B \subset C$, $C \in \mathcal{C} \Rightarrow C \in \mathcal{F}$ because if $\mu(C) \in H$ then $\mu(A_1 - C) \notin H$ and $\mu(C - B) \notin H$, which contradicts the choice of $\{A_1,\ldots,A_p\}$; (iii) $B \in \mathcal{F}$, $C \in \mathcal{F} \Rightarrow B \cap C \in \mathcal{F}$ because if $\mu(B \cap C) \in H$ then $\mu(B - B \cap C) \notin H$ and $\mu(C - B \cap C) \notin H$ and $\{B - B \cap C,\ C - B \cap C,\ A_2,\ldots,A_p\}$ gives another element of $\mathcal{P}$ containing $p + 1$ elements; (iv) $B \subset A_1$, $B \in \mathcal{C} \Rightarrow B \in \mathcal{F}$ or $A_1 - B \in \mathcal{F}$ and exactly one of them $\in \mathcal{F}$; and (v) $\varnothing \notin \mathcal{F}$.

Thus $\mathcal{F}$ is a maximal filter in the field $A_1 \cap \mathcal{C}$ on A_1. Let us define μ' on $A_1 \cap \mathcal{C}$ by

$$\begin{aligned} \mu'(C) &= \mu(A_1) && \text{if } C \subset A_1, C \in \mathcal{F}, \\ &= 0 && \text{if } C \subset A_1, C \in \mathcal{C} \text{ and } C \notin \mathcal{F}. \end{aligned}$$

Then μ' is a charge on $A_1 \cap \mathcal{C}$ and for any $A \in \mathcal{C}$, $A \subset A_1$, $\mu(A) - \mu'(A)$ equals $\mu(A - A_1)$ or $\mu(A)$ according as $\mu(A) \notin H$ or not. By (iv) this implies $\mu(A) - \mu'(A) \in H$ for all $A \in \mathcal{C}$, $A \subset A_1$, or $\mu - \mu'$ is an H-valued charge on $A_1 \cap \mathcal{C}$ with finite range. Hence by Case (i), $\mu - \mu'$ can be extended to an H-valued charge τ_1 on $A_1 \cap \mathcal{B}$. On the other hand, by extending the ultrafilter $\mathcal{F}$ in $A_1 \cap \mathcal{C}$ to an ultrafilter $\mathcal{F}'$ in $A_1 \cap \mathcal{B}$ and by defining τ_2 on $A_1 \cap \mathcal{B}$ by $\tau_2(A) = \mu(A_1)$ if $A \in \mathcal{F}'$ and $\tau_2(A) = 0$ if $A \notin \mathcal{F}'$, $A \in A_1 \cap \mathcal{B}$, we obtain a charge on $A_1 \cap \mathcal{B}$ extending μ'. Now $\tau_1 + \tau_2$ is a G-valued charge on $A_1 \cap \mathcal{B}$ extending $\mu|_{A_1 \cap \mathcal{C}}$.

Since this procedure can be adopted to each of the A_i's, we obtain our result.

REMARK 6. The extension we have obtained in Theorem 6 also has finite range.

ACKNOWLEDGEMENTS. The authors thank the referee for improving the original draft. Remark 4 grew out of a query of the referee.

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