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ANALYTICITY PRESERVING PROPERTIES OF RESOLVENTS FOR DEGENERATE DIFFUSION OPERATORS IN ONE DIMENSION

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ABSTRACT. Let $L = a(x)(d^2/dx^2) + b(x)(d/dx) + c(x)$ be a diffusion operator on a compact interval $I = [r_0, r_1]$ investigated by S. N. Ethier. Here, assume that a, b and c are real analytic functions on I, $a(x) > 0$ for $x \in (r_0, r_1)$, $a(r_i) = 0 \leq (-1)^i b(r_i)$ $(i = 0, 1)$, and both r_i $(i = 0, 1)$ are simple zeros of $a(x)$. It is shown that the resolvent $\{G_\lambda\}$ for L has the analyticity preserving property for sufficiently large λ, so that the equation $(L - \lambda)u = f$ is solvable in the space of real analytic functions on I. Some examples are given to show that the condition on L is best possible.

1. Introduction. The purpose of this paper is to study the analyticity preserving properties of resolvents for the diffusion operators investigated by Ethier in [**1**]. This is connected with the solvability of second order linear ordinary differential equations in the space of real analytic functions.

Let $I = [r_0, r_1]$ be a compact interval in $(-\infty, \infty)$ and $C^2(I)$ be the space of twice continuously differentiable real functions on I. Let L be a diffusion operator defined on $C^2(I)$:

$$L = a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx} + c(x),$$

where $a, b, c \in C^\infty(I)$, $a \geq 0$ and $a(r_i) = 0 \leq (-1)^i b(r_i)$ $(i = 0, 1)$. Here, $C^\infty(I)$ denotes the space of infinitely differentiable real functions on I. Let $C(I)$ be the space of continuous real functions on I. Ethier's result contains that an extension of L generates a unique strongly continuous nonnegative semigroup $\{T_t\}$ on $C(I)$ and $T_t\colon C^\infty(I) \to C^\infty(I)$ for $t \geq 0$. However, it will be shown that the resolvent $G_\lambda = \int_0^\infty e^{-\lambda t} T_t\, dt$ does not always inherit this property from the semigroup (see Examples 1, 2), so that in such cases $(L - \lambda)(C^\infty(I)) \neq C^\infty(I)$. Denote by $C^\omega(I)$ the space of real analytic functions on I. We will give a best possible condition under which $G_\lambda\colon C^\omega(I) \to C^\omega(I)$ for sufficiently large λ, so that $(L - \lambda)(C^\omega(I)) = C^\omega(I)$.

It should be noticed that L does not have the analytic hypoellipticity on I, because it has singular points (see Komatsu [**3**, Theorem 1]).

2. Ethier's result. Our result is based on the definitive result of Ethier on the differentiability preserving properties of the semigroups. We quote part of his result in the form used later. Let $C^0(I) = C(I)$ and $C^n(I)$ be the space of nth continuously differentiable real functions on I $(n = 1, 2, \ldots)$. For $n = 0, 1, \ldots$, $C^n(I)$ is the Banach space with the norm $\|\ \|_n$, $\|f\|_n = \sum_{k=0}^n \|f^{(k)}\|_0$, where $\|\ \|_0$ is the supremum norm.

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For the coefficients a, b, c of L, set

$$d_{kj} = \binom{k}{j-2} a^{(k-j+2)} + \binom{k}{j-1} b^{(k-j+1)} + \binom{k}{j} c^{(k-j)} \qquad (0 \le j \le k \le n),$$

where $\binom{k}{-2} = \binom{k}{-1} = 0$.

Ethier's result (see [**1**, Theorem 1]): *For* $n = 1, 2, \ldots$,

$$T_t \colon C^n(I) \to C^n(I) \qquad (t \ge 0)$$

and

$$||T_t||_n \le \exp(\lambda_n t) \qquad (t \ge 0),$$

where

$$\lambda_n = \max_{0 \le j \le n} \sum_{k=j}^{n} ||d_{kj}||_0.$$

3. Main result. We set the following assumption on the coefficients of L.

ASSUMPTION. (1) $a, b, c \in C^\omega(I)$,
(2) $a(x) > 0$ for $x \in I^0 = (r_0, r_1)$,
(3) $a(r_i) = 0 \le (-1)^i b(r_i)$ $(i = 0, 1)$,
(4) both r_i $(i = 0, 1)$ are simple zeros of $a(x)$.

THEOREM. *Under the assumption,*

$$G_\lambda \colon C^\omega(I) \to C^\omega(I) \quad for\, \lambda > \lambda_2.$$

First, we show the following lemma.

LEMMA. *Suppose the assumption. If* $\lambda \ne c(r_i)$ $(i = 0$ *or* $1)$, *then for any* $f \in C^\omega(I)$ *the equation* $(L - \lambda)u = f$ *has a real analytic solution* u *in a neighborhood of* r_i $(i = 0$ *or* $1)$.

PROOF. Assume that $i = 0$. By the assumption on L, we can write

$$a(x) = (x - r_0)\tilde{a}(x),$$

where $\tilde{a}(x) > 0$ $(r_0 \le x < r_1)$ and $\tilde{a}(x) \in C^\omega(I)$. Therefore, in a small neighborhood of r_0, the equation $(L - \lambda)u = f$ is equivalent to the equation

$$(x - r_0)u'' + \tilde{b}(x)u' + \tilde{c}(x)u = \tilde{f}(x), \tag{3.1}$$

where $\tilde{b}(x) = b(x)/\tilde{a}(x)$, $\tilde{c}(x) = \{c(x) - \lambda\}/\tilde{a}(x)$ and $\tilde{f}(x) = f(x)/\tilde{a}(x)$. Since $\tilde{c}(r_0) \ne 0$ and $\tilde{b}(r_0) \ge 0$, there is a formal power-series solution of (3.1) at $x = r_0$. The formal power-series solution becomes an actual solution of (3.1), because r_0 is a regular singular point of the operator $(x - r_0)(d^2/dx^2) + \tilde{b}(x)(d/dx) + \tilde{c}(x)$ (cf. [**2**, Theorem 6.1]).

PROOF OF THEOREM. Let f be a real analytic function on I. From Ethier's result, it follows that for $\lambda > \lambda_2$, $G_\lambda f \in C^2(I)$ and $u = G_\lambda f$ is a solution of $(L - \lambda)u = -f$ on I. Since $a(x) > 0$ for all $x \in I^0$, $L - \lambda$ has the analytic hypoellipticity on I^0 (see [**3**, Theorem 1]). Therefore, $G_\lambda f \in C^\omega(I^0)$ for $\lambda > \lambda_2$. Suppose that $i = 0$. If $\lambda > \lambda_2$, then $\lambda \ne c(r_0)$, because $\lambda_2 > \lambda_0$. Therefore, the equation $(L - \lambda)u = -f$ has a real analytic solution u_0 in a neighborhood of r_0.

Hence, for some $\delta > 0$, $u - u_0$ is a C^2-solution of the homogeneous equation

$$(L - \lambda)v = 0 \tag{3.2}$$

in $(r_0, r_0+\delta)$. The equation (3.2) is equivalent to the following equation in $(r_0, r_0+\delta)$:

$$(x - r_0)^2 v'' + (x - r_0)P(x)v' + Q(x)v = 0, \tag{3.3}$$

where $P(x) = (x - r_0)b(x)/a(x)$, $Q(x) = (x - r_0)^2\{c(x) - \lambda\}/a(x)$. By the assumption of the theorem, $P(r_0) \geq 0$ and $Q(r_0) = 0$. Thus the indicial equation for (3.3) relative to $x = r_0$, i.e., $\rho(\rho - 1) + P(r_0)\rho + Q(r_0) = 0$ has 0 and $1 - P(r_0)$ as roots. Therefore, a fundamental system of solutions $\{v_1(x), v_2(x)\}$ of (3.3) is given in the following form:

(i) if $\rho_1 \equiv 1 - P(r_0) = 0$ or 1, then

$$v_1(x) = (x - r_0)^{\rho_1} \sum_{n=0}^{\infty} a_n(x - r_0)^n,$$

$$v_2(x) = Av_1(x) \log(x - r_0) + \sum_{n=0}^{\infty} b_n(x - r_0)^n,$$

where a_n, A, b_n are real numbers, $a_0 \neq 0$, and $\sum a_n(x - r_0)^n$ and $\sum b_n(x - r_0)^n$ converge in a neighborhood of r_0;

(ii) if $\rho_1 \neq 0, 1$, then

$$v_1(x) = (x - r_0)^{\rho_1} \sum_{n=0}^{\infty} a_n(x - r_0)^n, \quad v_2(x) = \sum_{n=0}^{\infty} c_n(x - r_0)^n,$$

where a_n and c_n are real numbers, $a_0 \neq 0$, $c_0 \neq 0$, and $\sum a_n(x - r_0)^n$, $\sum c_n(x - r_0)^n$ converge in a neighborhood of r_0. Assume that δ is sufficiently small. Then, there are constants k_1 and k_2 such that

$$u(x) - u_0(x) = k_1 v_1(x) + k_2 v_2(x) \qquad (r_0 < x < r_0 + \delta).$$

In the case (i) with $A = 0$, $u(x)$ is real analytic at r_0. If $A \neq 0$, then

$$\begin{aligned} &k_2(x - r_0)^{\rho_1} \log(x - r_0) \\ &\qquad = \{u(x) - u_0(x) - k_1 v_1(x) - k_2 \textstyle\sum b_n(x - r_0)^n\}/A \sum a_n(x - r_0)^n. \end{aligned}$$

The right side is a C^2-function on $[r_0, r_0 + \delta)$, but the left side is not. Therefore, $k_2 = 0$, so that $u(x) = u_0(x) + k_1 v_1(x)$ is real analytic at r_0. In the case (ii) we can prove that $u(x)$ is real analytic at r_0 in the same way. The function u is also analytic at r_1. Consequently, $u \in C^\omega(I)$.

4. Examples. In this section, it is shown that the consequence of the theorem is not always valid if the condition (2) of the assumption is replaced by the condition $a(x) \geq 0$ $(x \in I)$ or the condition (4) is omitted.

Let us consider a linear ordinary differential equation with analytic data. When this equation has a C^∞-solution in an interval, there is a formal power-series solution of the equation at each point of the interval. Further assume that all the singular points of the equation are regular. Then in the same way as the proof of the theorem, it is shown that the C^∞-solution is a C^ω-solution in the interval.

Let L be a second order linear ordinary differential equation with polynomial coefficients and $u = \sum_{n=0}^{\infty} u_n(x - x_0)^n$ be a formal solution of $(L - \lambda)u = f$ with

analytic function f. Then the sequence $\{n(n-1)u_n\}$ satisfies a difference equation of Poincaré's type. Hence, the following examples are proved by the fundamental result of Perron in [**5**].

EXAMPLE 1. Let $L = x(x-1)^2(d^2/dx^2)$ and $I = [0,1]$. Then $G_\lambda\colon C^\omega(I) \not\to C^\omega(I)$ for every $\lambda > 0$, so that $G_\lambda\colon C^\infty(I) \not\to C^\infty(I)$ for every $\lambda > 0$.

EXAMPLE 2. Let $L = x(x-1)^2(3-x)(d^2/dx^2)$ and $I = [0,3]$. Then the same consequence as in Example 1 is obtained.

EXAMPLE 3. Let $L = x(x-1)(x-2)(d^2/dx^2)$. If $I \subset (-\infty, 2)$, then, $(L-\lambda)(C^\omega(I)) = C^\omega(I)$ for every $\lambda > 0$, and if $0, 1, 2 \in I$, then $(L-\lambda)(C^\omega(I)) \neq C^\omega(I)$ for every λ.

We prove only Example 1. The others are proved in a similar way. Set $f = (x-1)^2$. Suppose that $u = G_\lambda f \in C^\omega(I)$. Then $(L-\lambda)u = -f$, so that u can be continued analytically into the whole complex plane, because all the singular points of $L-\lambda$ are 0 and 1. Therefore the radius of convergence of $u(x) = \sum_{n=0}^\infty u_n(x-1)^n$ is infinity. Let $v_n = n(n-1)u_n$. Then the sequence $\{v_n\}$ satisfies

$$(1-\lambda/2)v_2 = -1, \tag{4.1}$$

$$\{1-\lambda/n(n-1)\}v_n - v_{n-1} = 0 \qquad (n \geq 3). \tag{4.2}$$

If $\lambda = n(n-1)$ $(n = 2, 3, \ldots)$, then the sequence $\{v_n\}$ does not exist. This is contradiction. Otherwise, the difference equation (4.2) is of Poincaré's type and its characteristic equation is $z - 1 = 0$. Therefore, from the result of Perron [**5**, Satz 3] (cf. [**4**, p. 548]), it follows that there is a fundamental solution $\{v'_n\}$ of (4.2) satisfying $\limsup_{n\to\infty} \sqrt[n]{|v'_n|} = 1$. Since $v_n \not\equiv 0$, $v_n = cv'_n$ for some constant $c \neq 0$. Hence $\limsup_{n\to\infty} \sqrt[n]{|u_n|} = 1$. This is contradiction. Consequently, for every λ, $G_\lambda f$ is not real analytic in I.

REMARK. Let us consider a second order linear ordinary differential equation with real analytic coefficients. Suppose that the equation has n (≥ 3) singular points (multiple singular points being counted according to their multiplicity) in an interval. Then, from the above examples, it is seen that the equation is not always solvable in the space of real analytic functions on the interval, even if it is locally solvable in this interval.

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