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A TECHNIQUE FOR PROVING UNIFORMITY

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ABSTRACT. We introduce a general technique for showing that if various properties hold, then they hold uniformly, assuming some determinacy. *Uniform* means in the boldface sense, that is, there are continuous functions taking indices to indices. For example, if $\mathbf{\Gamma}$ is a parametrized pointclass then all closure properties of $\mathbf{\Delta}$ are uniform, and the separation property for $\mathbf{\Gamma}$ implies uniform separation.

Our notation and terminology is that of Moschovakis [**5**]. Let $\mathbf{\Gamma}$ be an $\mathbf{R}$-parametrized (boldface) pointclass, closed under continuous preimages. Any pointclass which has a parametrization (that is, a universal set) also has a *good* parametrization, i.e. one for which there are continuous S-m-n functions [**5**, 3H.1]. In this note we will be concerned with the question of whether properties hold uniformly with respect to a given good parametrization. Given any two good parametrizations, there is a uniform way to go from an index for a set with respect to one parametrization to an index for the same set with respect to the other. So without loss of generality, we may assume our parametrization has particular properties that happen to be convenient. We make two such assumptions: We work with 2^ω rather than $\mathbf{R}$ or ω^ω, and we assume our S-m-n functions are not merely continuous, but in fact are Lipschitz functions, that is, strategies for II in a Gale-Stewart game (cf. [**8**]). Let $G^n_\alpha \subset (2^\omega)^n$ be the n-ary relation parametrized by $\alpha \in 2^\omega$. We call α an *index* for this relation. An S-m-n function is, of course, a function f of $m+1$ variables such that $f(\alpha, \beta_1, \ldots, \beta_m)$ is an index for the n-ary relation $\{(x_1, \ldots, x_n)\colon G^{m+n}_\alpha(\beta_1, \ldots, \beta_m, x_1, \ldots, x_n)\}$.

It is a well-known consequence of the S-m-n property that if $\mathbf{\Gamma}$ has some closure property, then it has it uniformly (see [**5**, 3H.2]). For example, if $\mathbf{\Gamma}$ is closed under $(\exists x \in 2^\omega)$ then there is a continuous function $\mathbf{u}\colon 2^\omega \to 2^\omega$ such that for all $\alpha \in 2^\omega$, for all $y \in 2^\omega$:

$$G^1_{\mathbf{u}(\alpha)}(y) \Leftrightarrow (\exists x \in 2^\omega)(G^2_\alpha(x, y)).$$

The technique to be described here allows us to prove, assuming AD (actually just $\mathbf{\Gamma}'$-determinacy, where $\mathbf{\Gamma}'$ is not much bigger than $\mathbf{\Gamma}$), that the same is true of $\mathbf{\Delta}$ ($\mathbf{\Delta} = \mathbf{\Gamma} \cap \check{\mathbf{\Gamma}}$, where $\check{\mathbf{\Gamma}} = \{A\colon$ the complement A is in $\mathbf{\Gamma}\}$), that is, if $\mathbf{\Delta}$ is closed under something, then it is *uniformly* closed. Code $\mathbf{\Delta}$ sets by elements of 2^ω as follows: α is an *index* for a $\mathbf{\Delta}$ subset of $(2^\omega)^n$ iff $G^n_{(\alpha)_0}$ is the complement of $G^n_{(\alpha)_1}$, where $\alpha \mapsto ((\alpha)_0, (\alpha)_1)$ is the pairing function of [**5**, p. 40]; an index α encodes the $\mathbf{\Delta}$ relation $D^n_\alpha = G^n_{(\alpha)_0} \subset (2^\omega)^n$. We will illustrate the technique with an example of a proof of uniform closure: If $\mathbf{\Delta}$ is closed under $(\exists x \in 2^\omega)$ then it is uniformly closed.

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A pointclass $\mathbf{\Gamma}$ is said to be *closed under intersection with closed sets* if for any sets A and F, if A is in $\mathbf{\Gamma}$ and F is closed then $A \cap F$ is in $\mathbf{\Gamma}$. Any class which contains the Boolean algebra generated by the open sets, and is closed under continuous preimages, is closed under intersection with closed sets (Steel [**7**]).

THEOREM (AD). *Let $\mathbf{\Gamma}$ be a parametrized pointclass with coding as described above. Suppose $\mathbf{\Gamma}$ and $\check{\mathbf{\Gamma}}$ are both closed under intersection with closed sets. If $\mathbf{\Delta}$ is closed under $(\exists x \in 2^\omega)$, then there is a continuous (in fact, Lipschitz) function $\mathbf{u}\colon 2^\omega \to 2^\omega$ such that for all $\alpha \in 2^\omega$, if α is the index of a $\mathbf{\Delta}$ binary relation then $\mathbf{u}(\alpha)$ is the index of a $\mathbf{\Delta}$ unary relation and for all $y \in 2^\omega$:*

$$D^1_{\mathbf{u}(\alpha)}(y) \Leftrightarrow (\exists x \in 2^\omega)(D^2_\alpha(x,y)).$$

PROOF. Consider the game on 2 in which I plays $\alpha \in 2^\omega$ and II plays $\beta \in 2^\omega$, each playing one integer at a time. II wins iff:

$$\begin{aligned}&[\alpha \text{ is not an index for a } \mathbf{\Delta} \text{ binary relation}] \quad \text{or}\\ &[(\alpha \text{ is an index for } D^2_\alpha \subset (2^\omega)^2) \,\&\, (\beta \text{ is an index for } D^1_\beta \subset 2^\omega)\\ &\qquad \& \, [(\forall y \in 2^\omega)(D^1_\beta(y) \Leftrightarrow (\exists x \in 2^\omega)(D^2_\alpha(x,y)))]].\end{aligned}$$

A winning strategy for II gives the desired continuous function; so it is enough to show that II wins the game.

Suppose not, and let σ be a winning strategy for I. Let R be the "range" of σ, that is,

$$\begin{aligned}R(\alpha) \Leftrightarrow &\text{ There is a run of the game in which I plays according to}\\ &\text{ the strategy } \sigma \text{ and I plays } \alpha.\end{aligned}$$

R is the continuous image of a compact set, hence compact. Because σ is a winning strategy, all α's in R are actually indices for $\mathbf{\Delta}$ relations. Let

$$Q(\alpha,x,y) \Leftrightarrow R(\alpha) \,\&\, D^2_\alpha(x,y).$$

That is,

$$Q(\alpha,x,y) \Leftrightarrow R(\alpha) \,\&\, G^2_{(\alpha)_0}(x,y) \Leftrightarrow R(\alpha) \,\&\, \neg G^2_{(\alpha)_1}(x,y).$$

Since $\mathbf{\Gamma}$ and $\check{\mathbf{\Gamma}}$ are both closed under intersection with closed sets, Q is both $\mathbf{\Gamma}$ and $\check{\mathbf{\Gamma}}$, hence $\mathbf{\Delta}$. Let

$$S(\alpha,y) \Leftrightarrow (\exists x \in 2^\omega) Q(\alpha,x,y).$$

By hypothesis, $\mathbf{\Delta}$ is closed under $(\exists x \in 2^\omega)$, so S is in $\mathbf{\Delta}$. For any α, let $S_\alpha(y) \Leftrightarrow S(\alpha,y)$. Since there exists a Lipschitz S-m-n function for $\mathbf{\Gamma}$, there is a Lipschitz function $\mathbf{s}\colon 2^\omega \to 2^\omega$ such that for all α, $\mathbf{s}(\alpha)$ is a $\mathbf{\Delta}$-index for $S_\alpha \subset 2^\omega$.

Consider the run of the game in which I plays by σ and produces some $\alpha \in 2^\omega$, and II plays according to $\mathbf{s}$ and produces a $\beta \in 2^\omega$ which is a $\mathbf{\Delta}$-index for S_α. This α is in R, so for all $y \in 2^\omega$:

$$\begin{aligned}D^1_\beta(y) &\Leftrightarrow S_\alpha(y) \Leftrightarrow S(\alpha,y) \Leftrightarrow (\exists x \in 2^\omega) Q(\alpha,x,y)\\ &\Leftrightarrow (\exists x \in 2^\omega)[R(\alpha) \,\&\, D^2_\alpha(x,y)] \Leftrightarrow (\exists x \in 2^\omega) D^2_\alpha(x,y).\end{aligned}$$

By definition of the game, II wins, a contradiction. $\square$

I would like to thank Yiannis Moschovakis for simplifying my original proof, by pointing out that S-m-n functions can be taken to be Lipschitz. The idea of having

one player in a game play by an S-m-n function has been used for other purposes by Louveau [**3**].

There is clearly nothing special about this particular closure property; this method will establish that any reasonable closure property of $\mathbf{\Delta}$ is uniform. One such property deserves special mention. Let n be odd and let $\kappa < \boldsymbol{\delta}^1_n$. By a theorem of Martin [**5**, 7D.9], assuming AD, $\mathbf{\Delta}^1_n$ is closed under quantification of the form $(\exists \xi < \kappa)$, and therefore, by the coding lemma, is closed under unions of length κ. Formally,

Let $\varphi\colon P \to \kappa$ be a $\mathbf{\Delta}^1_n$-norm of length κ and let $Q(x,y)$ be φ-invariant on x, that is, if x, $x' \in P$ and $\varphi(x) = \varphi(x')$ and $Q(x,y)$ then

$(*)$ $Q(x',y)$. (Thus Q is essentially a relation $\hat{Q} \subset \kappa \times 2^\omega$.) Let

$$R(y) \Leftrightarrow (\exists x \in P) Q(x,y) \Leftrightarrow (\exists \xi < \kappa)\hat{Q}(\xi, y).$$

If Q is $\mathbf{\Delta}^1_n$, then R is also $\mathbf{\Delta}^1_n$.

The technique used in the proof of the above theorem shows that $(*)$ holds uniformly, that is, there is a continuous function taking $\mathbf{\Delta}^1_n$-indices for invariant relations $Q(x,y)$ to $\mathbf{\Delta}^1_n$-indices for $(\exists \xi < \kappa)\hat{Q}(\xi, y)$. (For $n = 3$ this is due to Kechris and Martin [**2**]; for $n > 3$ it was previously unknown.)

This method of proof can be used to show the uniformity of structure properties other than closure properties, both for structure properties of $\mathbf{\Delta}$ (such as uniformization) and for those of $\mathbf{\Gamma}$. For example, the S-m-n Theorem implies that if $\mathbf{\Gamma}$ has the reduction property then it has uniform reduction [**5**, 4B.14], but it does not do the same for the separation property. This method shows that assuming AD, separation implies uniform separation, at least for pointclasses closed under intersection with closed sets.

REMARK (KECHRIS [**1**]). Kechris has proved in ZF + DC that a class $\mathbf{\Gamma}$ and its dual $\check{\mathbf{\Gamma}}$ cannot both have uniform separation, provided that $\mathbf{\Gamma}$ is closed under $\exists^\omega$, $\forall^{\mathbf{R}}$, and continuous preimages. This gives a new proof of a weak form of Van Wesep's theorem [8] that AD implies $\mathbf{\Gamma}$ and $\check{\mathbf{\Gamma}}$ cannot both have separation. Harrington has proved that if ZFC is consistent then it is consistent with ZFC that both $\mathbf{\Sigma}^1_3$ and $\mathbf{\Pi}^1_3$ have separation.

There are other types of coding besides that of universal sets, and hence other types of uniformity. For example, the uniform Suslin-Kleene Theorem (see [**5**, 7B]) says that there is a continuous function taking a $\mathbf{\Delta}^1_1$-index for a set to a Borel code for the same set. The usual proof of this (due to Kleene) actually gives a recursive function. According to [**5**, p. 400 and **6**], there was no known proof that does not use the Recursion Theorem. However Borel codes have the S-m-n property, by [**5**, 7B.1], and therefore the above technique yields a new proof of the uniform Suslin-Kleene Theorem, one which does not use the Recursion Theorem. This proof is, however, inferior to the Kleene proof for two reasons: First, the continuous function need not be recursive, and secondly because $\mathbf{\Pi}^1_1$-determinacy is needed. (The game in question is a difference of $\mathbf{\Pi}^1_1$ sets. The determinacy of all such games is equivalent to $\mathbf{\Pi}^1_1$-determinacy which is equivalent to the existence of sharps [**4**].)

If $\mathbf{\Gamma}$ is a parametrized pointclass, then any property of $\mathbf{\Gamma}$ which holds uniformly, will automatically also hold (uniformly) in lightface (ω-parametrized) classes Γ such that $\mathbf{\Gamma}$ is the usual boldface class associated with Γ (as in [**5**, p. 183]), and such that Γ satisfies the following two properties: The universal set for $\mathbf{\Gamma}$ is in Γ, and the continuous function which computes the uniformity is a Δ-real. For example, applying this fact to $(*)$ we have that, assuming AD, for fixed $\kappa < \delta^1_n$ (n odd), there is an x_0 such that for all x, if $x_0 \leq_T x$, then $\Delta^1_n(x)$ is closed under quantification of the form $(\exists \xi < \kappa)$. (For $n = 3$ this is due to Kechris and Martin [**2**].)

References

1. A. S. Kechris, personal communication.
2. A. S. Kechris and D. A. Martin, *On the theory of* Π^1_3 *sets of reals*, Bull. Amer. Math. Soc. **84** (1978), 149–151.
3. A. Louveau, *Borel sets and the analytical hierarchy*, Proc. Herbrand Sympos., Logic Colloq. '81 (J. Stern, ed.), North-Holland, Amsterdam, 1982, pp. 209–215.
4. D. A. Martin, *Borel and projective games* (to appear).
5. Y. N. Moschovakis, *Descriptive set theory*, North-Holland, Amsterdam, 1980.
6. ——, Lecture notes, Cornell Recursion Theory Conf., 1982.
7. J. R. Steel, *Closure properties of pointclasses* (Cabal Seminar, 1977–1979), Lecture Notes in Math., vol. 839 (A. S. Kechris, D. A. Martin and Y. N. Moschovakis, eds.), Springer-Verlag, Berlin and New York, 1981, pp. 147–163.
8. R. Van Wesep, *Wadge degrees and descriptive set theory* (Cabal Seminar, 1976–1977), Lecture Notes in Math., vol. 689 (A. S. Kechris and Y. N. Moschovakis, eds.), Springer-Verlag, Berlin and New York, 1978, pp. 151–170.

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