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PROPERTY C AND FINE HOMOTOPY EQUIVALENCES

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ABSTRACT. We show that within the class of metric σ-compact spaces, proper fine homotopy equivalences preserve property C, which is a slight generalization of countable dimensionality. We also give an example of an open fine homotopy equivalence of a countable dimensional space onto a space containing the Hilbert cube.

1. Introduction. In this note we shall study the behaviour of some "dimensionality properties" of infinite-dimensional spaces under fine homotopy equivalences. Let us recall that a map $f\colon X \to Y$ is a *fine homotopy equivalence* if for every open cover $\mathcal{U}$ of Y there exists a map $g\colon Y \to X$ such that $f \circ g$ is $\mathcal{U}$-homotopic to id_Y and $g \circ f$ is $f^{-1}(\mathcal{U})$-homotopic to id_X. Let us mention that a closed map $f\colon X \to Y$ of an ANR X onto an ANR Y is a fine homotopy equivalence if: (a) all fibers of f are contractible or (b) f is a cell-like map, i.e. f is a proper map with fibers of trivial shape (see **[Ha1** and **To]**). We are interested in countable dimensional spaces (a space X is *countable dimensional* if X is a countable union of finite dimensional sets) and spaces having property C (a metric space X has property C, abbreviated $X \in C$, iff given any sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ of positive real numbers, there exists an open cover $\mathcal{U}$ of X such that $\mathcal{U} = \bigcup_{n=1}^{\infty} \mathcal{U}_n$, where $\mathcal{U}_n$ is a pairwise disjoint family with $\mathrm{diam}(U) < \varepsilon_n$ for every $U \in \mathcal{U}_n$, $n \in \mathbf{N}$). Note that each metric, countable dimensional space has property C and that a space containing a topological copy of the Hilbert cube $Q = [-1,1]^{\infty}$ does not have property C (for details see **[Ha2]**).

Because fine homotopy equivalences do not raise finite dimension, the following question was posed by D. Henderson and G. Kozlowski.

Question 1. *Do cell-like maps, which are fine homotopy equivalences, preserve countable dimension*?

In this note we will show that within the class of σ-compact spaces, *proper* fine homotopy equivalences preserve property C and we give an example of an *open* fine homotopy equivalence α of the space $\sigma = \{(x_i) \in l_2\colon x_i = 0$ for all but finitely many $i\}$ onto the space $\Sigma = \{(x_i) \in l_2\colon \sum_{i=1}^{\infty}(ix_i)^2 < \infty\}$. The map $\alpha\colon \sigma \to \Sigma$ "raises" dimension because σ is countable dimensional but Σ contains the Hilbert cube Q and hence $\Sigma \notin C$.

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2. The main result. In this section we formulate and prove our main result.

2.1. THEOREM. *Let X be a σ-compact metric space with property C and let $f\colon X \to Y$ be a proper fine homotopy equivalence of X onto a metric space Y. Then $Y \in C$.*

PROOF. Because every space which is the countable union of compacta with property C, has property C itself, it is enough to prove that each compact subset of Y has property C. Let A be a compact subset of Y and let $B = f^{-1}(A)$. Let ρ be an extension on Y of a given metric on A. Define a compatible metric d on X by the formula $d(x_1,x_2) = \delta(x_1,x_2) + \rho(f(x_1),f(x_2))$, where δ is a compatible metric on X and $x_1,x_2 \in X$. Observe that $\rho(f(x_1),f(x_2)) \leqslant d(x_1,x_2)$ for every $x_1,x_2 \in X$. Now choose a sequence $\{\varepsilon_n\}_1^\infty$ of positive real numbers. Since $X \in C$, there is an open cover $\mathcal{V}$ of X such that $\mathcal{V} = \bigcup_{n=1}^{\infty}\mathcal{V}_n$, where $\mathcal{V}_n$ is a pairwise disjoint family consisting of sets of diameter less than $\varepsilon_n/3$. Because B is compact we can choose a finite subfamily $\mathcal{V}'$ of $\mathcal{V}$ which covers B. Let $n_0 = \min\{n \in \mathbf{N}\colon \mathcal{V}' \subset \bigcup_{m\leqslant n}\mathcal{V}_m\}$ and let $W = \bigcup\mathcal{V}'$. Then $f(W)$ is a neighborhood of A in Y. Let $g\colon Y \to X$ be a map such that $\rho(f\circ g, \mathrm{id}_Y) < \eta$, where $\eta = \frac{1}{3}\circ\min\{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_{n_0}\}$, and $g(A) \subset W$. Then $\mathcal{U} = g^{-1}(\mathcal{V}')$ is a cover of A. We will show that the cover $\mathcal{U}$ has the properties required in the definition of property C for the sequence $\{\varepsilon_n\}_1^{n_0}$ and the metric ρ. To this end, first observe that $\mathcal{U} = \bigcup_{n=1}^{n_0} g^{-1}(\mathcal{V}_n \cap \mathcal{V}')$ and that $g^{-1}(\mathcal{V}_n \cap \mathcal{V}')$ is a pairwise disjoint family for $n = 1,2,\ldots, n_0$. Let $V \in \mathcal{V}_n \cap \mathcal{V}'$. We shall prove that $\operatorname{diam}_\rho g^{-1}(V) < \varepsilon_n$. Take $y_1,y_2 \in g^{-1}(V)$ and for $i = 1,2$ let $x_i = g(y_i)$. Then

$$\begin{aligned}\rho(y_1,y_2) &\leqslant \rho(y_1,fg(y_1)) + \rho(fg(y_1),fg(y_2)) + \rho(y_2,fg(y_2))\\ &< 2\eta/3 + \rho(f(x_1),f(x_2))\\ &\leqslant 2\eta/3 + d(x_1,x_2) < 2\eta/3 + \varepsilon_n/3 < \varepsilon_n.\end{aligned}$$

We conclude that $\operatorname{diam}_\rho g^{-1}(V) < \varepsilon_n$. Observe that the cover $\mathcal{U}' = \mathcal{U} \cup \varnothing$ has the properties required in the definition of property C for the sequence $\{\varepsilon_n\}_{n=1}^\infty$ and the metric ρ. $\square$

REMARK. In the proof of the theorem we used only the fact that the map is approximately right invertible, i.e. given an open cover $\mathcal{U}$ of Y there exists a map $g\colon Y \to X$ such that $f\circ g$ is $\mathcal{U}$-close to id_Y.

G. Kozlowski **[Ko]** proved that a proper map $f\colon X \to Y$ between ANR's is a fine homotopy equivalence iff f is a hereditary shape equivalence, i.e., $\mathrm{Sh}(f^{-1}(A)) = \mathrm{Sh}(A)$ for each compact set A in Y. This result is used in the proof of the following

2.2. COROLLARY. *Let X be a σ-compact space with property C and let $f\colon X \to Y$ be a hereditary shape equivalence. Then Y has property C.*

PROOF. Without losing generality, we can assume that X and Y are compact. By the Freudenthal Expansion Theorem, see e.g. Borsuk **[Bo]**, X is the inverse limit of finite dimensional ANR's, say $X = \varprojlim\{X_n,f_n\}$, with each X_n an ANR. Let M be the infinite mapping cylinder of the sequence $\{X_n,f_n\}$ with a copy of X attached at its end. Then $M \in \mathrm{ANR}$ and $M \in C$ (observe that we added a countable dimensional set to X). Let $\mathcal{G}_f = \{f^{-1}(y)\colon y \in Y\} \cup \{\text{points}\}$, then $\mathcal{G}_f$ is a cell-like decomposition of M. Let $p_f\colon M \to M/\mathcal{G}_f$ be the quotient map. Because f is a hereditary shape equivalence, $M/\mathcal{G}_f \in$ ANR and p_f is a fine homotopy equivalence **[Ko]**. By Theorem 2.1, $M/\mathcal{G}_f \in C$ and since Y embeds in $M/\mathcal{G}_f$, $Y \in C$. $\square$

3. The example. In this section we construct an example of an open fine homotopy equivalence of σ onto Σ.

3.1. EXAMPLE. There exists a map $\alpha: \sigma \to \Sigma$ such that:

(1) α is "onto",

(2) α is open,

(3) point inverses of α are homeomorphic to σ,

(4) α is a fine homotopy equivalence.

PROOF. Let $\beta: K \to Q$ be an *open* map of the universal Menger curve K onto the Hilbert cube such that $\beta^{-1}(q)$ is homeomorphic to K for each $q \in Q$ (see **[An]**). Let 2^K_f and 2^Q_f denote the hyperspaces of finite subsets of K and Q, respectively. By **[Cu]**, 2^K_f is homeomorphic to σ and 2^Q_f is homeomorphic to Σ. Let $\alpha: 2^K_f \to 2^Q_f$ be the map defined by $\alpha(\{k_1, k_2, \ldots, k_n\}) = \{\beta(k_1), \beta(k_2), \ldots, \beta(k_n)\}$. Then α satisfies (1)–(4). The conditions (1) and (2) are satisfied because the map α is open and onto. We will check (3). Take distinct $q_1, q_2, \ldots, q_n \in Q$, arbitrarily. Observe that

$$\alpha^{-1}(\{q_1, q_2, \ldots, q_n\}) = \{A_1 \cup A_2 \cup \cdots \cup A_n : A_i \subset \beta^{-1}(q_i) \text{ is finite and nonempty}\}$$
$$\approx \underbrace{2^K_f \times 2^K_f \times \cdots \times 2^K_f}_{n} \approx \sigma^n \approx \sigma.$$

It is not hard to check that α is a UV^∞-map, i.e., given $y \in \Sigma$ and a neighborhood U of y, there is a neighborhood $V \subset U$ of y such that $\alpha^{-1}(V)$ is contractible in $\alpha^{-1}(U)$. By **[Ha1]**, α is a fine homotopy equivalence.

3. Questions. At the end of this note, we state some open problems.

Question 2. *Let $f: X \to Y$ be a* closed *fine homotopy equivalence such that $X \in ANR$. If $X \in C$, does it follow that $Y \in C$?* This is true for σ-compact X.

Question 3. *Let $f: W \to V$ be an affine map of a σ-compact convex subset $W \subset l_2$ onto a subset $V \subset l_2$. If $W \in C$, does it follow that $V \in C$?*

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