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A GENERALIZATION OF ALTMAN'S ORDERING PRINCIPLE

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ABSTRACT. A maximality principle on quasi-ordered quasi-metric spaces, containing, in particular, Altman's and the Brézis-Browder ordering principles, is given. As an application, a local mapping theorem extending—from a "functional" viewpoint—a similar one due to Altman, is derived.

Let X be a nonempty set, and let $\leqslant$ be a *quasi-ordering* (that is, a reflexive and transitive relation) on X. Concerning existence of certain "maximal" elements in the structure $(X, \leqslant)$, the following Altman's ordering principle [**3**] must be considered as a starting point of our developments.

THEOREM 1. *Let the quasi-ordered set $(X, \leqslant)$ be such that every descending sequence has a lower bound, and let F: $X^2 \to R$ be a function satisfying $F(x,y) \leqslant 0$ when $x \leqslant y$, as well as:*

(i) *for any y in X, $F(.,y)$ is bounded below on $X(y, \geqslant) = \{t \in X; y \geqslant t\}$;*

(ii) *for any x in X, $F(x,.)$ is decreasing;*

(iii) $\limsup_{n\to\infty} F(x_{n+1}, x_n) = 0$, *for every descending sequence $(x_n;\ n \in N)$ in X. Then, to any x in X, there corresponds z in X with* (a) $x \geqslant z$, *and* (b) $y \leqslant z$ *implies* $F(y,z) = 0$.

As already pointed out by this author, Theorem 1 may be viewed as a natural and straightforward extension of the well-known Brézis-Browder ordering principle [**4**, **6**], a result that found a number of interesting applications to convex as well as nonconvex analysis (see the above references); consequently, a generalization of it might be of some avail. It is precisely our main aim to formulate a statement of this kind; as a direct application, a generalization of a basic mapping theorem due to Altman [**2**] will be given.

Let $(X, \leqslant)$ be a quasi-ordered set. A function d: $X^2 \to [0, \infty)$ will be termed *$\leqslant$-quasi-metric* when $d(x,x) = 0$ for all x in X, and

(iv) *for every $\varepsilon > 0$ there exists $\delta > 0$ such that $x \leqslant y \leqslant z$ and $d(x,y)$, $d(x,z) < \delta$ imply $d(y,z) < \varepsilon$.*

In such a context, a sequence $(x_n;\ n \in N)$ in X will be said to be *d-asymptotic* provided that $\liminf_{n\to\infty} d(x_n, x_{n+1}) = 0$ and an element z in X will be termed *d-maximal* in case $z \leqslant y$ implies $d(z,y) = 0$. Now, under these conventions, the main

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result of the present note is

THEOREM 2. *Suppose the structure* $(X, \leqslant)$ *satisfies*

(v) *any ascending sequence has an upper bound*

and let the $\leqslant$*-quasi-metric d on X be such that*

(vi) *any ascending sequence is d-asymptotic.*

Then, to every x in X there corresponds a d-maximal element z in X with $x \leqslant z$.

PROOF. As a matter of fact, the proof of Theorem 2 is implicit in [7] but, for the sake of completeness, we shall, however, indicate its basic lines. Letting x in X, it follows—by the same procedure as in the author's paper we already mentioned—that an ascending sequence $(x_n;\ n \in N)$ in X may be found with $x \leqslant x_n$, $n \in N$, and

$$(1) \qquad y \in X, n \in N, x_n \leqslant y \quad \text{imply} \quad d(x_n,y) < 2^{-n}.$$

By (v), $x_n \leqslant z$, $n \in N$, for some z in X. Let y in $X(z, \leqslant) = \{t \in X;\ z \leqslant t\}$ and let $\varepsilon > 0$ be arbitrarily fixed. There exists, by (1), an $n(\delta) \in N$ with $d(x_n,z)$, $d(x_n,y) < \delta$, $n \geqslant n(\delta)$ (where δ is that appearing in (iv)) and this gives $d(z,y) < \varepsilon$, ending the argument. Q.E.D.

Now, clearly, Altman's result follows from Theorem 2 if we replace $\leqslant$ by its dual and put $d(x,y) = |F(y,x)|$, $x,y \in X$ (note at this moment the boundedness hypothesis (i) may be dropped in the statement of Theorem 1, while (ii) may be rephrased as

(ii)′ $F(x,y) \geqslant F(x,z) + F(y,z)$ *when* $x \leqslant y \leqslant z$).

On the other hand, in case d is a *quasi-metric* (i.e., a nonsufficient metric) on X, Theorem 2 reduces to Theorem 1 of the above quoted author's paper, which turns out to be a generalization of the Brézis-Browder ordering principle distinct from that indicated by Altman. Moreover, calling the element z in X, $\leqslant$*-maximal* provided that $z \leqslant y$ implies $y \leqslant z$, it is evident that, under the supplementary hypothesis

(vii) $x \leqslant y$ *and* $d(x,y) = 0$ *imply* $y \leqslant x$,

Theorem 2 yields a Zorn maximality principle in $(X, \leqslant)$ (for any x in X there exists a $\leqslant$-maximal element z in X with $x \leqslant z$). A more specific proof of this fact may be obtained along the following lines. Let C be a $\leqslant$*-chain* in X; constructing as before an ascending sequence $(x_n;\ n \in N)$ in C satisfying (1) (modulo C) the following possibilities hold; either $(x_n;\ n \in N)$ is cofinal in C and then, any upper bound of it is clearly an upper bound of C, or $(x_n;\ n \in N)$ is not cofinal in C in which case, any upper bound (in C) of this sequence is, by (vii), an upper bound of C. So, any $\leqslant$-chain in X is bounded above and the classical Zorn theorem applies. Of course, in case (vii) does not hold, Theorem 2 (hence, in particular, Altman's and/or the author's ordering principles we quoted before) cannot be viewed as a Zorn maximality principle in $(X, \leqslant)$. Finally, a slight generalization of our main result could be obtained if, instead of d, a *denumerable family* D of $\leqslant$ -quasi-metrics on X were used. More specifically, assume (vi) is fulfilled for any d in D; then, under the acceptance of (v), to every x in X there corresponds z in X with (a)′ $x \leqslant z$, and (b)′ $z \leqslant y$ implies $d(z,y) = 0$, $d \in D$; an implicit proof of this fact may be found in [**8**].

Now, passing to the second part of our developments, let $(X,d, \leqslant)$ be a quasi-ordered metric space, and $(Y, \|\cdot\|)$ a normed space. Under the terminology of [7], let us call $\leqslant$ *self-closed* when the limit of any ascending convergent sequence is an upper bound of it, and the ambient space $\leqslant$-*complete* in case any ascending Cauchy sequence is a convergent one. Also, given the mapping T from $D(T) \subset X$ into Y, let us term it $\leqslant$-*closed* provided, if the ascending sequence $(x_n; n \in N)$ in $D(T)$ satisfies $x_n \to x$ and $Tx_n \to y$ for some x in X and y in Y, respectively, then $x \in D(T)$ and $Tx = y$. Suppose in what follows, X is a $\leqslant$-complete metric space under the self-closed quasi-ordering $\leqslant$, and Y is a Banach space under its norm. As an interesting application of the main result, the following "local" mapping theorem can be stated and proved.

THEOREM 3. *Suppose the $\leqslant$-closed mapping T: $D(T) \to Y$ and the element x_0 in $D(T)$ with $Tx_0 \neq 0$ are such that a number q in $[0,1)$ and a couple of functions b,f: $(0,\infty) \to [0,\infty)$ satisfying*

$$(2) \qquad (1 - s/t)b(t) \leqslant f(t) - f(s), \qquad t,s > 0,\ qt \leqslant s < t,$$

may be found with the property: for any x in $D_0(T) = \{y \in D(T);\ x_0 \leqslant y,\ d(x_0,y) \leqslant f(\|Tx_0\|)/(1-q)\}$ with $Tx \neq 0$ there exist $x' \geqslant x$ in $D(T)$ and ε in $(0,1]$ with

$$(3) \qquad \|Tx' - (1-\varepsilon)Tx\| \leqslant q\varepsilon\|Tx\|, \qquad d(x,x') \leqslant \varepsilon b(\|Tx\|).$$

Then, the equation $Tx = 0$ has at least a solution in $D_0(T)$.

PROOF. Suppose by contradiction that the assertion of the theorem were not valid; so, given any x in $D_0(T)$ there exists $x' \geqslant x$ in $D(T)$ such that (3) holds. Observe that, as an immediate consequence,

$$\|Tx'\| \leqslant (1 - (1-q)\varepsilon)\|Tx\|, \qquad \|Tx - Tx'\| \leqslant r(\|Tx\| - \|Tx'\|)$$

where $r = (1+q)/(1-q)$. Putting $t = \|Tx\|$, $s = (1-(1-q)\varepsilon)t$, we clearly have $qt \leqslant s < t$ so, by (2) plus (3) (the second part), we get (denoting $g(t) = f(t)/(1-q)$, $t > 0$)

$$(4) \qquad d(x,x') \leqslant \varepsilon b(t) \leqslant (1 - s/t)b(t)/(1-q) \leqslant g(t) - g(s).$$

Now, let $E_0(T)$ denote the subset of all y in $D_0(T)$ with $d(x_0,y) \leqslant g(\|Tx_0\|) - g(\|Ty\|)$ and observe that $x \in E_0(T)$ plus (4) implies $x_0 \leqslant x'$ and

$$d(x_0,x') \leqslant g(\|Tx_0\|) - g(s) \leqslant g(\|Tx_0\|)$$

whence $x' \in D_0(T)$. It follows at once that $0 < \|Tx'\| \leqslant s$ and therefore, by (4) again (observe that, as a consequence of (2), g is increasing on its existence domain),

$$d(x,x') \leqslant g(\|Tx\|) - g(\|Tx'\|),$$

proving $x' \in E_0(T)$ too; moreover, as $Tx \neq Tx'$, we necessarily have $x \neq x'$. Let e denote the "product" metric on $D(T)$,

$$e(x,y) = \max(d(x,y), \|Tx - Ty\|), \qquad x,y \in D(T),$$

and let $\leqq$ indicate the ordering on $D_0(T)$ defined as

$$x \leqq y \text{ if and only if } x \leqslant y,\ d(x,y) \leqslant g(\|Tx\|) - g(\|Ty\|) \text{ and } \|Tx - Ty\| \leqslant r(\|Tx\| - \|Ty\|).$$

Our main aim is to demonstrate that conditions (v)–(vii) are fulfilled in $(E_0(T), e, \leqq)$ and this will lead us to the desired contradiction. (Indeed, it would follow then by the main result that, for the given x_0 in $E_0(T)$, a $\leqq$-maximal element z in $E_0(T)$ may be found with $x_0 \leqq z$; on the other hand, by the above relations, a $z' \in E_0(T)$ may be chosen so that $z \leqq z'$ and $z \neq z'$, showing z is not maximal in $(E_0(T), \leqq)$). To this end, let $(x_n;\ n \in N)$ be an ascending (modulo $\leqq$) sequence in $E_0(T)$, that is,

$$\text{(5)} \qquad \begin{cases} x_n \leqslant x_m, \quad d(x_n, x_m) \leqslant g(\|Tx_n\|) - g(\|Tx_m\|) \text{ and} \\ \|Tx_n - Tx_m\| \leqslant r(\|Tx_n\| - \|Tx_m\|), \qquad n, m \in N, n \leqslant m. \end{cases}$$

As an easy consequence of this fact, $(g(\|Tx_n\|);\ n \in N)$ and $(\|Tx_n\|;\ n \in N)$ are descending sequences in $[0, \infty)$ —hence Cauchy sequences—and therefore, $(x_n;\ n \in N)$ is an ascending (modulo $\leqslant$) d-Cauchy sequence in X and $(Tx_n;\ n \in N)$ a Cauchy sequence in Y. By our completeness/closedness hypothesis, $x_n \xrightarrow{d} x$ and $Tx_n \to Tx$ for some x in $D(T)$, proving $x_n \xrightarrow{e} x$ and establishing the e-asymptotic property (vi). Moreover, as $D_0(T)$ is closed with respect to ascending (modulo $\leqslant$) d-convergent sequences, it clearly follows that $x \in D_0(T)$ (hence $Tx \neq 0$) in which situation, observing that, as a consequence of (5),

$$d(x_n, x_m) \leqslant g(\|Tx_n\|) - g(\|Tx\|), \qquad n, m \in N, n \leqslant m,$$

and

$$\|Tx_n - Tx_m\| \leqslant r(\|Tx_n\| - \|Tx\|), \qquad n, m \in N, n \leqslant m,$$

we immediately derive (performing the limit as m tends to infinity) $x_n \leqq x$, $n \in N$ (and consequently, $x \in E_0(T)$) proving (v) and completing the argument, since (vii) is trivially satisfied in our case. Q.E.D.

Concerning the basic hypothesis (3), let X be a Banach space and let X_+ be a closed *quasi-cone* in X (that is, $\lambda x + \mu y \in X_+$ whenever $\lambda, \mu \geqslant 0$ and $x, y \in X_+$); then, indicating by $\leqslant$ the usual quasi-ordering induced by X_+, the following condition appears as a sufficient one for its validity:

(viii) *for any x in $D_0(T)$ with $Tx \neq 0$ there exists v in X_+ with $dT_x(v) = -Tx$ and $\|v\| \leqslant b(\|Tx\|)$*

(here, dT_x indicates the usual Gâteaux derivative of the mapping T at the point x); indeed, putting $x' = x + \varepsilon v$, (3) becomes

$$\|(T(x + \varepsilon v) - Tx)/\varepsilon - dT_x(v)\| \leqslant q\|Tx\|,$$

a relation that clearly takes place for sufficiently small ε in $(0, 1]$. Note that such a condition will be fulfilled when, e.g., a $\delta > 0$ may be found with

$$-Tx/\|Tx\| \in dT_x(S(0, \delta) \cap X_+)$$

for all x in $D_0(T)$ with $Tx \neq 0$ (where $S(0, \delta)$ denotes the closed sphere with center $0 \in X$ and radius δ; see also Cramer and Ray [5]). On the other hand, a sufficient condition (of a practical interest) for (2) is

(ix) *$t \vdash b(t)$ is increasing and $c(t) = \int_0^t (b(s)/s)\,ds < \infty$, $t > 0$,*

because, letting $f(t) = c(t/q)$, $t > 0$, we have, for any couple $t, s > 0$, $qt \leqslant s < t$,

$$(1 - s/t)b(t) \leqslant (b(u)/u)((t/q) - (s/q)), \qquad s/q \leqslant u \leqslant t/q,$$

proving our assertion; of course, under the assumptions that $\leqslant$ is the trivial quasi-ordering on X and b is continuous, the particular case just considered yields Altman's Theorem 1.1 stated in [**2**] (cf. also [**5**, Theorem 2.1]). It is useful to notice at this moment that an alternative condition with respect to (ix) is

(x) $t \vdash b(t)/t$ *is decreasing and* $c(t) = \int_0^t (b(s)/s)\,ds < \infty$, $t > 0$,

showing that our mapping theorem effectively includes that of Altman, because, e.g., the function $b(t) = t^{1/2}e^{-t}$, $t > 0$, satisfies (x) but not (ix). A partial extension of these considerations to mappings taking values in a Fréchet space may be found in Turinici [**9**] (see also [**8**]). Some useful applications of Theorem 2 (under its particular variants just discussed) to concrete functional equations were indicated by Altman [**1**].

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