THE GELFAND SUBALGEBRA OF REAL OR NONARCHIMEDEAN VALUED CONTINUOUS FUNCTIONS

JESUS M. DOMINGUEZ

Abstract. Let \( L \) be either the field of real numbers or a nonarchimedean rank-one valued field. For topological space \( T \) we study the Gelfand subalgebra \( C_\delta(T, L) \) of the algebra of all \( L \)-valued continuous functions \( C(T, L) \). The main result is that if \( T \) is a paracompact locally compact Hausdorff space, which is ultraregular if \( L \) is nonarchimedean, then for \( f \in C(T, L) \) the following statements are equivalent: (1) There exists a compact set \( K \subseteq T \) such that \( f(T - K) \) is finite, (2) \( f \) has finite range on every discrete closed subset of \( T \), and (3) \( f \in C_\delta(T, L) \).

Throughout this paper \( L \) will stand for either the valued field \( \mathbb{R} \) or a nonarchimedean rank-one valued field, and \( T \) for a Hausdorff topological space. \( T \) will be assumed completely regular in the real case and ultraregular in the nonarchimedean case.

We will denote by \( C(T, L) \), or simply by \( C \) if there is no confusion, the algebra over \( L \) consisting of all \( L \)-valued continuous functions on \( T \), and by \( C_K \) the ideal of those continuous \( f \in C \) with compact support. Let \( \mathfrak{M} \) be the set of maximal ideals of \( C \), and denote by \( C_0 \) the Gelfand subalgebra of \( C \), consisting of all \( f \in C \) with the property that, for each \( M \in \mathfrak{M} \), there exists \( \lambda \in L \) such that \( (f - \lambda) \in M \). (The concept of Gelfand subalgebras of general algebras has been introduced by N. Shell —formerly N. Shilkret—in [6].) We will denote by \( C_F \) the subalgebra of \( C \) consisting of those \( f \in C \) for which there exists a compact set \( K \subseteq T \) such that \( f(T - K) \) is finite.

**Proposition 1.** For any \( f \in C_0 \), \( f(T) \) is compact.

**Proof.** First, we prove that \( f(T) \) is a precompact set.

**Real case.** It suffices to see that \( f \) is bounded, and this follows from [5, 5.7(b)].

**Nonarchimedean case.** Take \( \varepsilon > 0 \). Note that any two (closed-open) \( \varepsilon \)-radius spheres \( B(\alpha) = \{ \mu \in L \mid |\mu - \alpha| < \varepsilon \} \) are equal or disjoint. Choose an indexed set \( (\alpha_i)_{i \in I} \) in \( f(T) \) such that \( B(\alpha_i)_{i \in I} \) is disjoint and covers \( f(T) \). We claim \( I \) is finite. In fact, assume, to the contrary, that \( I \) is infinite. Let \( A_I = \bigcup_{j \neq i} f^{-1}(B(\alpha_j)) \). Since the family of closed-open sets \( (A_i)_{i \in I} \) has the finite intersection property, there exists \( M \in \mathfrak{M} \) such that \( A_i \subseteq Z[M] \) for any \( i \in I \). On the other hand, since \( f \in C_0 \) there exists \( \lambda \in L \) such that \( (f - \lambda) \in M \) and hence \( Z(f - \lambda) \cap A_i \neq \emptyset \) for any \( i \in I \), which is a contradiction. Thus \( I \) is finite and so \( f(T) \) is precompact.
Now we will prove that \( f(T) \) is compact. Endow \( \mathbb{M} \) with the Zariski topology (also called the Stone topology) and identify any \( t \in T \) with the fixed maximal ideal \( M_t = \{ f \in C \mid f(t) = 0 \} \). Then, by virtue of the Gelfand-Kolmogoroff theorem and its ultraregular analogue (see [5, 7.3] and [1]), \( \mathbb{M} \) can be considered as the Stone-Cech compactification of \( T \) in the real case and as the Banaschewski one in the nonarchimedean case. Since \( f(T) \) is a precompact set, \( f \) can be uniquely extended to a continuous function \( f^\beta: \mathbb{M} \to L \). Since \( f \in C_0 \), for each \( M \in \mathbb{M} \) one has that \( f^\beta(M) = \lambda \) if \( (f - \lambda) \in M \); hence \( (f - f^\beta(M)) \in M \) and \( Z(f - f^\beta(M)) \neq \emptyset \), so for each \( M \in \mathbb{M} \) there exists \( t \in T \) such that \( f^\beta(M) = f(t) \). This shows that \( f^\beta(\mathbb{M}) \subset f(T) \), and obviously \( f(T) \subset f^\beta(\mathbb{M}) \), thus \( f(T) = f^\beta(\mathbb{M}) \) is a compact set.

**Proposition 2.** \( C_F \subset C_0 \).

**Proof.** Take \( f \in C_F \) and let \( K \) be a compact set such that \( f(T - K) = \{ \lambda_1, \ldots, \lambda_n \} \). If \( M = M_\lambda \) is a fixed maximal ideal of \( C \) then \( (f - f(t)) \in M \); if \( M \) is a free maximal ideal of \( C \) then \( (f - f(t)) \in C_K \subset M \), so there exists \( 1 \leq i \leq n \) such that \( (f - \lambda_i) \in M \). Hence \( f \in C_0 \) and so \( C_F \subset C_0 \).

**Theorem.** Assume that \( T \) is paracompact and locally compact. Then for \( f \in C \) the following statements are equivalent: (1) \( f \in C_F \), (2) \( f \) has finite range on every discrete closed subset of \( T \), and (3) \( f \in C_0 \).

**Proof.** It is evident that (1) \( \Rightarrow \) (2). From Proposition 2 it follows that (1) \( \Rightarrow \) (3). Now let \( f \in C \) and assume that \( f(T - K) \) is infinite for every compact set \( K \subset T \), which implies, in particular, that \( T \) is not compact. We claim that there exists a discrete closed subset of \( T \) in which \( f \) has infinite range and that \( C_F \subset C_0 \). Note that the theorem follows from this claim. For the proof of the claim we will distinguish two cases.

**Real case.** First, we will consider the case in which \( T \) is \( \sigma \)-compact (see Bourbaki [3, p. 68]). Take a sequence \((U_n)\) of relatively compact open sets such that \( U_n \subset U_{n+1} \) and \( T = \bigcup U_n \). From the assumptions on \( f \) there exists a sequence \((t_n)\), \( t_n \in U_{i_n} - \bar{U}_{i_n-1} \) for some increasing sequence \((i_n)\), such that \( f(t_n) \neq f(t_m) \) for \( n \neq m \). For convenience, we set \( V_n = U_{i_n} \). It is evident that the set \( \{ t_n \mid n \in \mathbb{N} \} \) is a discrete closed subset of \( T \) on which \( f \) has infinite range. To see that \( f \notin C_0 \) set \( K_n = V_{3n} - V_{3n-1} \), \( L_n = V_{3n-2} \). Since \( K_n \) is compact, \( L_n \) is closed, \( K_n \cap L_n = \emptyset \) and \( T \) is completely regular, there exists a continuous function \( g_n: T \to [0,1] \) such that \( g_n|_{K_n} = 0 \) and \( g_n|_{L_n} = 1 \). On the other hand, there exists another continuous function \( l_n: \mathbb{R} \to [0,1] \) such that \( f(l_n) = Z(l_n) \) and so \( Z(f - f(l_n)) \cap Z(g_n) = Z(h_n) \) where \( h_n = \sup (l_n \circ f, g_n) \). Now set \( D_k = \bigcup_{k \leq n} Z(h_n) \) and \( d_k = \inf_{k \leq n} h_n \). For any \( m \in \mathbb{N} \) there exists \( i(m) \in \mathbb{N} \) such that \( h_n|_{V_m} = 1 \) if \( n > i(m) \), so \( d_k|_{V_m} = \inf_{k \leq n<i(m)} h_n \). Hence \( d_k \subset C \) and \( Z(d_k) = D_k \). Since the family of all \( z \)-sets \( D_k \) has the finite intersection property, there exists \( M \in \mathbb{M} \) such that \( D_k \subset Z(M) \) for any \( k \in \mathbb{N} \). If \( f \in C_0 \) then \( (f - \lambda) \in M \) for some \( \lambda \in \mathbb{R} \) and consequently \( Z(f - \lambda) \cap D_k \neq \emptyset \) for all \( k \), which is a contradiction. Thus \( f \notin C_0 \) and the claim is proved for \( T \sigma \)-compact.
For general $T$ the proof is reduced to the above case if we show that $T$ contains a compact closed-open set $T'$ with the property that $f(T - K)$ is infinite for every compact $K \subset T'$. Since $T$ is a paracompact locally compact Hausdorff space, $T$ is the disjoint union of a family $(T_i)_{i \in I}$ of $\sigma$-compact open subsets of $T$. If some $T_i$ has the above stated property, set $T' = T_i$. Otherwise, take a sequence $(t_n)$ such that $t_n \in T_i$ and $f(t_n) \neq f(t_m)$ for $n \neq m$, and set $T' = \bigcup T_i$. This completes the proof of the real case.

Nonarchimedean case. From the topological assumptions on $T$ it follows that $T$ is the disjoint union of a family $(T_i)_{i \in I}$ of compact-open subsets of $T$. From the assumptions on $f$ there exists $t_n \in T_i$, $n = 1, 2, \ldots$, such that $i_n \neq i_m$ and $f(t_n) \neq f(t_m)$ for $n \neq m$. Since the range of $f$ is infinite over the discrete closed subset $\{t_n | n \in \mathbb{N}\}$, the proof will be completed if we show $f \not\in C_0$. Define

$$h_k(t) = \begin{cases} f(t) - f(t_n), & t \in T_i \text{ and } n \geq k, \\ 1, & \text{otherwise}. \end{cases}$$

Then $h_k$ is continuous, and, by letting $D_k = Z(h_k)$, we may proceed as in the real case.

The hypothesis of paracompactness is not superfluous as the following example shows:

Example 1 (see [5, p. 123]). Let $\omega_1$ be the first uncountable ordinal and let $W^*$ be the set of all ordinals less than $\omega_1 + 1$ endowed with the interval topology. Let $T^* = W^* \times N^*$, where $N^*$ denotes the one-point compactification $N \cup \{w\}$ of $N$, and let $T = T^* - \{t\}$, where $t = (\omega_1, w)$. $T$ is a pseudocompact locally compact Hausdorff space. Since $T$ is pseudocompact we have $C_0(T, \mathbb{R}) = C(T, \mathbb{R})$. The continuous function $f$ defined by $(\alpha, n) \to 1/n$, $(\alpha, w) \to 0$ belongs to $C_0(T, \mathbb{R})$, but $f \not\in C_F(T, \mathbb{R})$.

The above example shows that, in general, the equality $C_K = \bigcap \{M \in \mathfrak{M} \mid M \text{ is free}\}$ does not hold (see [5, p. 123]). However, as a consequence of our theorem one has the following

**Corollary.** If the space $T$ is paracompact and Hausdorff locally compact then $C_K = \bigcap \{M \in \mathfrak{M} \mid M \text{ is free}\}$.

**Proof.** If $f$ lies in every free maximal ideal of $C$ then $f \in C_0$, and according to the theorem one has $f \in C_F$. Let $K$ be a compact subset of $T$ such that $f(T - K) = \{\lambda_1, \ldots, \lambda_n\}$. From [5, p. 58], it follows that $Z(f - \lambda_i)$ is compact for $\lambda_i \neq 0$, $1 \leq i \leq n$. Hence $f \in C_K$.

**Remark** (see [2, p. 20]). The Theorem and the Corollary are also true if $L$ is replaced by the valued field of complex numbers $\mathbb{C}$. This can be easily deduced from the fact that the maximal ideals of $C(T, \mathbb{R})$ are in 1-1 correspondence with the maximal ideals of $C(T, \mathbb{C})$ under the mapping $M \to M + iM$. (The inverse of this map is the map sending the maximal ideal $m$ of $C(T, \mathbb{C})$ into re$(m)$, where re$(m)$ denotes the collection of real parts re$(G)$ of functions $G$ in $m$.)

Finally, we give an example in which $T$ is not a normal space but, nevertheless, the conclusion of the Theorem is true.
Example 2 (see [5, Exercise 8L]). Let \( w_1 \) and \( W^* \) be as in Example 1. Let \( T^* = W^* \times W^* \) and \( T = T^* - \{(w_1, w_1)\} \). \( T \) is a pseudocompact locally compact Hausdorff space, \( T^* \) is the one-point compactification of \( T \) and every function in \( C(T, \mathbb{R}) \) is constant on a deleted neighbourhood of \((w_1, w_1)\). Hence, \( C(T, \mathbb{R}) = C_0(T, \mathbb{R}) = C_p(T, \mathbb{R}) \), but \( T \) is not normal.

I would like to thank the referee for his very valuable comments and suggestions.

References


Departamento de Algebra y Fundamentos, Facultad de Ciencias, Universidad de Valladolid, Valladolid, Spain.