PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 90, Number 1, January 1984

MINIMAX THEOREMS FOR ANR'S

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ABSTRACT. Some minimax theorems for real valued functions on a product space are proved. The functions considered are lower semicontinuous or continuous. Also, some related results on the intersection of subsets of a product space are proved.

1. Introduction. The objective of this paper is to prove some minimax theorems and some related results. One minimax theorem is the following

THEOREM. *Let* $\alpha = \sup_{x \in X} \min_{y \in Y} f(x, y)$ *and* $\beta = \min_{y \in Y} \sup_{x \in X} f(x, y)$. *Here* X *is an acyclic ANR,* Y *a compact finite dimensional ANR, and* $f\colon X \times Y \to R$ *a function satisfying the following conditions.*

(a) $f(x, \cdot)$ *is lower semicontinuous, all* x *in* X.

(b) $\{(x, y) \mid f(x, y) > \alpha\}$ *is open.*

(c) $\{x \in X \mid f(x, y) > \alpha\}$ *is contractible or empty, all* y *in* Y.

(d) $\{y \in Y \mid f(x, y) \leq \alpha\}$ *is acyclic, all* x *in* X.

Then $\alpha = \beta$.

Note that X need not be compact (cf. Ha [8]) and Y need not be contractible. Note, also, that if f is lower semicontinuous then (a) and (b) of the theorem are satisfied.

There are several minimax theorems in the literature with hypotheses similar to those above but with "contractible" and "acyclic" replaced by the more special "convex" (Sion [12], Fan [4, 5], Browder [2], Ha [8])—however, these convex type theorems do not require Y to be finite dimensional. In [3] Debreu proved a minimax theorem for continuous f with contractible and acyclic type hypotheses (see the corollary on p. 890 and the footnote on p. 892 of [3]). In [1] Bourgin generalized this to certain nonacyclic situations. In the present paper, Theorem 3.6 is a result for continuous f with somewhat different hypotheses than those of Debreu—in particular, one of the spaces is an arbitrary compact topological space.

In §2 basic terminology is given. In §3 the minimax theorems are proven. §4 relates multifunction composition and the intersection of n subsets of a product of n spaces.

2. Terminology. The spaces in the hypotheses of the theorems are assumed to be *nonempty*. R is the real numbers with the usual topology. ANR means ANR (metric). The reader may find it helpful to remember that a subset of Euclidean space is an

Received by the editors April 21, 1983.

1980 *Mathematics Subject Classification*. Primary 54H25, 47H10, 49B40.

[1]Research partially supported by a grant from the University of Kansas.

ANR iff it is locally contractible (Hu [9, p. 168]). A space is *acyclic* if it is nonempty, connected and has vanishing rational homology in all positive dimensions. Note that any contractible space is acyclic.

For a function $h: X \to R = (-\infty, +\infty)$ or $h: X \to [-\infty, +\infty]$, $\sup\{h(x) \mid x \in X\}$ and $\inf\{h(x) \mid x \in X\}$ will always exist since $-\infty$ and $+\infty$ are allowed in either case. "$\max\{h(x) \mid x \in X\}$ exists" means $h(x) \leq h(\bar{x})$ for some $\bar{x}$, all x; thus $\max(h)$ is finite unless h itself takes infinite values (and similarly for $\min(h)$). The following elementary facts are easily verified for $f: X \times Y \to R$.

(1) Suppose X and Y are compact, $f(x, \cdot)$ is lower semicontinuous, all x, and $f(\cdot, y)$ is upper semicontinuous, all y. Then $\max_{x \in X} \min_{y \in Y} f(x, y)$ and $\min_{y \in Y} \max_{x \in X} f(x, y)$ both exist.

(2) Suppose Y compact and $f(x, \cdot)$ lower semicontinuous for all x. Then $\sup_{x \in X} \min_{y \in Y} f(x, y)$ and $\min_{y \in Y} \sup_{x \in X} f(x, y)$ both exist.

(3) Suppose X compact and $f(\cdot, y)$ upper semicontinuous for all y. Then $\max_{x \in X} \inf_{y \in Y} f(x, y)$ and $\inf_{y \in Y} \max_{x \in X} f(x, y)$ both exist.

(4) $\sup_{x \in X} \inf_{y \in Y} f(x, y) \leq \inf_{y \in Y} \sup_{x \in X} f(x, y)$.

A *multifunction* $m: X \to Y$ assigns to each $x \in X$ a *nonempty* subset of Y. The graph of m is $G(m) = \{(x, y) \mid y \in m(x)\} \subset X \times Y$. m is an open graph [closed graph] multifunction if $G(m)$ is open [closed] in $X \times Y$. A (continuous) *selection* for the multifunction m is a single valued (continuous) function $f: X \to Y$ with $f(x) \in m(x)$, all $x \in X$. A *fixed point* for m is an $x \in X$ satisfying $x \in m(x)$. m is *compact* if $m(X)$ is contained in a compact subset of Y. m is *admissible* (Gorniewicz [6, p. 27]) if there exist a space Z and two single valued continuous functions $p: Z \to X$, $q: Z \to Y$ such that $q(p^{-1}(x)) \subset m(x)$, p is proper, and $p^{-1}(x)$ is acyclic for all x in X.

3. Minimax theorems. The theorems below require the following two propositions, neither stated in maximum generality.

3.1 PROPOSITION ([10, COROLLARY 2.2]). *Suppose that X is a compact finite dimensional ANR and $m: X \to Y$ is an open graph multifunction with contractible values. Then m has a continuous selection.*

3.2 PROPOSITION (Gorniewicz [6, COROLLARY 3.7]). *Let X be an acyclic ANR and $w: X \to X$ a compact admissible (see §2) multifunction. Then w has a fixed point.*

We first prove a theorem about the intersection of subsets of a product (cf. Fan [5]). If $S \subset X \times Y$ then let $S(x) = \{y \mid (x, y) \in S\}$ and $S^{-1}(y) = \{x \mid (x, y) \in S\}$.

3.3 THEOREM. *Suppose that X is an acyclic ANR and Y is a compact finite dimensional ANR. Suppose $S, T \subset X \times Y$ satisfy:*

(1) *S is open in $X \times Y$ and $S^{-1}(y)$ is contractible and nonempty, all $y \in Y$.*

(2) *T is closed in $X \times Y$ and $T(x)$ is acyclic and nonempty, all $x \in X$.*

Then $S \cap T$ is nonempty.

PROOF. Define $m: Y \to X$ by $m(y) = S^{-1}(y)$. Then m is a multifunction with contractible values and open graph (since $G(m) \equiv S$). Proposition 3.1 gives a continuous single valued $g: Y \to X$ with $g(y) \in m(y)$, all y in Y. Define $w: X \to X$

by $w(x) = gT(x)$. Since w factors through Y it is a compact multifunction. Define p: $T \to X$ by $p(x, y) = x$, and q: $T \to X$ by $q(x, y) = g(y)$. p is proper since Y is compact and T closed, $p^{-1}(x) = x \times T(x)$ is acyclic, and $qp^{-1}(x) = qT(x) = w(x)$. Thus (see §2) w is admissible and Proposition 3.2 shows that it has a fixed point, say $\bar{x} \in gT(\bar{x})$. Then $\bar{x} = g(\bar{y})$ with $\bar{y} \in T(\bar{x})$. So $\bar{x} \in S^{-1}(\bar{y})$ and thus $(\bar{x}, \bar{y}) \in S \cap T$, proving the theorem.

3.4 THEOREM. *Let*

$$\alpha = \sup_{x \in X} \min_{y \in Y} f(x, y) \quad \textit{and} \quad \beta = \min_{y \in Y} \sup_{x \in X} f(x, y).$$

Here X is an acyclic ANR, Y a compact finite dimensional ANR, and f: $X \times Y \to R$ a function satisfying the following conditions.

(a) *$f(x, \cdot)$ is lower semicontinuous, all x in X.*

(b) *$\{(x, y) \mid f(x, y) > \alpha\}$ is open.*

(c) *$\{x \in X \mid f(x, y) > \alpha\}$ is contractible or empty, all y in Y.*

(d) *$\{y \in Y \mid f(x, y) \leqslant \alpha\}$ is acyclic, all x in X.*

Then $\alpha = \beta$.

PROOF. By the comments of §2, α and β both exist (possibly infinite) and $\alpha \leqslant \beta$, so we need only prove $\alpha \geqslant \beta$.

Define $S = \{(x, y) \mid f(x, y) > \alpha\}$. Then S is open (by (b)) and $S^{-1}(y)$ is contractible or empty for all y (by (c)). Suppose, by way of contradiction, that $S^{-1}(y)$ is not empty, all y.

Define $T = \{(x, y) \mid f(x, y) \leqslant \alpha\}$. Then T is closed by (b) and $T(x)$ is acyclic for all x in X by (d). It follows easily from the definition of α that $T(x)$ is nonempty for all y in Y.

The hypotheses of the set intersection theorem are satisfied so $S \cap T$ must be nonempty. But this is not true so $S^{-1}(y)$ must be empty for some y, say $\bar{y}$. Then $f(x, \bar{y}) \leqslant \alpha$, all x, so $\sup_{x \in X} f(x, \bar{y}) \leqslant \alpha$, and $\beta \leqslant \alpha$, proving the theorem.

Small modifications of the proof yield the following version.

3.4′ THEOREM. *Let*

$$\alpha = \max_{x \in X} \inf_{y \in Y} f(x, y) \quad \textit{and} \quad \beta = \inf_{y \in Y} \max_{x \in X} f(x, y).$$

Here X is a compact finite dimensional ANR, and Y an acyclic ANR, and f: $X \times Y \to R$ a function satisfying the following conditions.

(a) *$f(\cdot, y)$ is upper semicontinuous, all y in Y.*

(b) *$\{(x, y) \mid f(x, y) < \beta\}$ is open.*

(c) *$\{y \in Y \mid f(x, y) < \beta\}$ is contractible or empty, all x in X.*

(d) *$\{x \in X \mid f(x, y) \geqslant \beta\}$ is acyclic, all y in Y.*

Then $\alpha = \beta$.

The statement of 3.4 requires exact knowledge of α (and 3.4′ requires exact knowledge of β). It is probably more realistic to assume (in 3.4) only approximate knowledge of α. Thus 3.4(b) would be rephrased as "$\{(x, y) \mid f(x, y) > t\}$ is open for t near α" with corresponding adjustment of 3.4(c) and (d).

Now a result for continuous f will be proved. First, recall that a function $h\colon X \to R$ is *quasiconcave* if $\{x \mid h(x) > t\}$ is convex for each t, and is *quasiconvex* if $\{x \mid h(x) < t\}$ is convex for each t. The next definition describes larger classes of functions.

3.5 DEFINITION. $h\colon X \to R$ is

(1) *t-upper acyclic* if $\{x \in X \mid h(x) > t\}$ is acyclic or empty,

(2) *t-lower acyclic* if $\{x \in X \mid h(x) < t\}$ is acyclic or empty.

3.6 THEOREM. *Let*

$$\alpha = \max_{x \in X} \min_{y \in Y} f(x, y) \quad \textit{and} \quad \beta = \min_{y \in Y} \max_{x \in X} f(x, y).$$

Suppose X is a compact Hausdorff space, Y a compact acyclic ANR, and $f\colon X \times Y \to R$ a continuous function satisfying:

(1) *$f(\cdot, y)$ is t-upper acyclic, all y in Y, all t near β,*

(2) *$f(x, \cdot)$ is t-lower acyclic, all x in X, all t near α.*

Then $\alpha = \beta$.

PROOF. By the comments in §2, α and β exist (possibly infinite) and $\alpha \leqslant \beta$, so we need only prove $\alpha \geqslant \beta$.

Let $\varepsilon > 0$, $S^{-1}(y) = \{x \in X \mid f(x, y) \geqslant \beta - \varepsilon\}$ and $T(x) = \{y \in Y \mid f(x, y) \leqslant \alpha + \varepsilon\}$. From the definition of α and β, S^{-1} and T have nonempty values. It follows from (1) and (2) and the following lemma that T and S^{-1} have acyclic values (the lemma is an easy consequence of the continuity of Čech homology).

3.7 LEMMA. *Let $h\colon X \to R$ be an upper [lower] semicontinuous function and $V(t) = \{x \in X \mid h(x) > t\}$ $[V(t) = \{x \in X \mid h(x) < t\}]$ and $H(t) = \{x \in X \mid h(x) \geqslant t\}$ $[H(t) = \{x \in X \mid h(x) \leqslant t]$. If X is compact and $V(t)$ is acyclic for all t in some open interval then $H(t)$ is acyclic for all t in that interval.*

Since f is continuous, S^{-1} and T have closed graph. Define $m = m_\varepsilon = TS^{-1}\colon Y \to Y$. Then, by Gorniewicz [**6**, Theorem 2.7], m_ε is admissible and compact. By 3.2 above, m_ε has a fixed point, say $(\bar{x}, \bar{y})$ in $S^{-1}(\bar{y}) \times T(\bar{x})$. Then $\bar{x} \in S^{-1}(\bar{y})$ so $f(\bar{x}, \bar{y}) \geqslant \beta - \varepsilon$, $\bar{y} \in T(\bar{x})$, so $f(\bar{x}, \bar{y}) \leqslant \alpha - \varepsilon$ and $\beta \leqslant \alpha + 2\varepsilon$ for all small ε so $\beta \leqslant \alpha$.

COMMENT. In [**7**] it is shown that Proposition 3.2 and, also the composition result from [**6**] used above are valid for ANE (compact)—a larger class of spaces than ANR's. Thus "ANR" in 3.4, 3.4′, 3.6 and 4.2, can be replaced by "ANE (compact)".

4. Composition and intersection. The composition method of §3 can be used to prove general set intersection theorems. In this section we first relate composition and intersection and then prove one such intersection result.

Let X_1, $1 \leqslant i \leqslant n$, be n spaces and $S_j \subset X_1 \times \cdots \times X_n$, $1 \leqslant j \leqslant n$, be subspaces. Define $X[i] = \pi\{X_j | j \neq i\}$, the projection π_j: $X[i] \to X_j$ and, for given multifunctions m_j: $A \to X_j, j \neq i$, define $m = (m_j)$: $A \to X[i]$. Define

$$S_i\colon X[i] \to X_i, \quad S_i(x_i, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) = \{x_i \in X_i | (x_1, \ldots, x_n) \in S_i\},$$

$$S_i'\colon X[i] \to X[i-1], \quad i \neq 1, \qquad (S_i')_j = \begin{cases} \pi_j, & 1 \leqslant j \leqslant i-2, i+1 \leqslant j \leqslant n, \\ S_i, & j = i, \end{cases}$$

$$S_1'\colon X[1] \to X[n], \qquad (S_1')_j = \begin{cases} \pi_j, & 2 \leqslant j \leqslant n-1, \\ S_1, & j = 1, \end{cases}$$

$$(*) \qquad X[n] \xrightarrow{S_n'} X[n-1] \to \cdots \to X[2] \xrightarrow{S_2'} X[1] \xrightarrow{S_1'} X[n].$$

4.1 THEOREM. *$S_1 \cap \cdots \cap S_n \neq \varnothing$ iff the composition $S = S_1' \cdots S_n'$ has a fixed point.*

PROOF. Check that $S(\bar{x}_1, \ldots, \bar{x}_{n-1}) = \bigcup\{S_1(x_2, \ldots, x_n) \times x_2 \times \cdots \times x_{n-1} | x_n \in S_n(\bar{x}_1, \ldots, \bar{x}_{n-1})$ and $x_j \in S_j(\bar{x}_1, \ldots, \bar{x}_{j-1}, x_{j+1}, \ldots, x_n)$, $j = n-1, \ldots, 2\}$. Then $(\bar{x}_1, \ldots, \bar{x}_{n-1})$ is a fixed point iff $(\bar{x}_1, \ldots, \bar{x}_{n-1}, \bar{x}_n) \in S_1 \cap \cdots \cap S_n$ where $(\bar{x}_1, \ldots, \bar{x}_{n-1}) \in S_1(\bar{x}_2, \ldots, \bar{x}_{n-1}, \bar{x}_n) \times \bar{x}_2 \times \cdots \times \bar{x}_n$.

4.2 THEOREM. *Let $X_1, \ldots, X_n$ be compact contractible finite dimensional ANR's. Suppose $S_j \subset X_1 \times \cdots \times X_n$, $j = 1, 2, \ldots, n$, are subsets which are either open or closed. Suppose for any given $x_1, \ldots, x_{i-1}$, $x_{i+1}, \ldots, x_n$, that the set $\{x_i \in X_i | (x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) \in S_i\}$ is contractible and nonempty. Then $S_1 \cap \cdots \cap S_n$ is not empty.*

PROOF. Each open S_i: $X[i] \to X_i$ has a selection by Proposition 3.1. This gives a selection for S_i'. Replace, in $(*)$, each S_i' by its selection so that $(*)$ is now a composition of closed graph multifunctions—which are admissible. By Gorniewicz [6] the composition is admissible, and by Proposition 3.2 it has a fixed point. Now 4.2 follows from 4.1.

Note that if all the S_i are closed then the hypotheses on the X_i and the values of the S_i can be weakened.

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