ALMOST-QUATERNION \((m - 1)\)-SUBSTRUCTURES ON \(S^{4m-3}\)

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Abstract. We prove that \(S^{4m-3}\) admits an almost-quaternion \((m - 1)\)-substructure if and only if \(m = 2\), completing the missing case in our paper On quaternionic James numbers and almost-quaternion substructures on the sphere.

In [2], an almost-quaternion substructure on an orientable \(n\)-manifold \(M\) is defined as the reduction of the structure group of \(T(M)\) from \(SO(n)\) to \(Sp(k) \times SO(n - 4k)\). This is equivalent to the existence of a \(4k\)-dimensional subbundle of \(T(M)\) together with two normalized almost complex structure maps \(F, G\) (see [1]), defined on the total space of the subbundle such that \(FG = -GF\). In [2] we have given a theorem which describes the values of \(k\) and \(n\) such that \(S^n\) admits an almost-quaternion \(k\)-substructure. There, all the cases were covered except for the case \(n = 4m - 3\) and \(k = m - 1\) for some \(m\). In this note we prove the following theorem about this case:

**Theorem.** \(S^{4m-3}\) admits an almost-quaternion \((m - 1)\)-substructure if and only if \(m = 2\).

**Proof.** First let \(m = 2\). Since \(\pi_4(U(2)/Sp(1)) = 0\), it follows that the fibration \(U(2)/Sp(1) \to U(3)/Sp(1) \to S^5\) admits a cross section. But then the fibration \(SO(5)/Sp(1) \to SO(6)/Sp(1) \to S^5\) admits a cross section proving that \(S^5\) admits an almost-quaternion 1-substructure.

Next assume \(m > 2\) and let us assume that \(S^{4m-3}\) admits an almost-quaternion \((m - 1)\)-substructure. Then there exists a \((4m - 4)\)-dimensional subbundle \(\xi\) of \(T(S^{4m-3})\) and normalized almost-complex substructure maps \(F, G\) defined on the total space \(E(\xi)\) such that \(FG = -GF\). Also, there exists an orthonormal 1-frame \(x \to w(x)\) such that the fiber of \(E(\xi)\) at \(x\) is orthogonal to \(w(x)\).

On the other hand let \(\eta\) be the subbundle of \(T(S^{4m-3})\) whose fiber at \(x\) is the orthogonal complement of the 1-frame \(x \to ix\) where \(i\) denotes the usual complex structure on \(\mathbb{R}^{4m-2}\). On the total space \(E(\eta)\) we will define two normalized almost complex substructures \(F_1, G_1\) such that \(F_1G_1 = -G_1F_1\).

Received by the editors January 28, 1983.

1980 Mathematics Subject Classification. Primary 55S40; Secondary 53C15.

Key words and phrases. Sectioning fiber spaces and bundles, almost complex, contact, symplectic, almost product structures.

I have just been informed by Hideaki Oshima that he had obtained another proof of the same result using homotopy exact sequences of certain fibrations.

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0002-9939/84 $1.00 + $.25 per page

155
Let $R(x)$ be the identity if $ix = w(x)$. Assume $ix \neq w(x)$. Let $\beta(x) = w(x) - (w(x) \cdot ix)ix$. Then $\{ix, \beta(x)/\|\beta(x)\|\}$ is an orthonormal basis for the subspace spanned by $ix$ and $w(x)$. Let $R(x)$ be the orthogonal transformation taking $ix$ to $w(x)$, $\beta(x)/\|\beta(x)\|$ to $-(w(x) \cdot \beta(x))ix/\|\beta(x)\| + (w(x) \cdot ix)\beta(x)/\|\beta(x)\|$, and leaving the orthogonal complement of the subspace spanned by $ix$ and $w(x)$ invariant. Note that the restriction of $R(x)$ to the subspace spanned by $ix$ and $w(x)$ is a rotation by the angle between $ix$ and $w(x)$.

Now, on the total space $E(\eta)$ we define normalized almost complex substructures $F_1, G_1$ by

$$F_1(v) = R(x)^{-1}(F(R(x)v)), \quad G_1(v) = R(x)^{-1}(G(R(x)v))$$

for all $x \in S^n$ and $v \in T_xS^n \cap E(\eta)$. Then clearly $F_1G_1 = -G_1F_1$.

We will show that for $m > 2$ this leads to a contradiction. To do this we slightly change the proof of Kirchoff’s theorem (Theorem 41.19 of [3]) as follows:

For each $y \in S^{4m-3}$ we define linear transformations $F(y): \mathbb{R}^{4m} \rightarrow \mathbb{R}^{4m}$ and $G(y): \mathbb{R}^{4m} \rightarrow \mathbb{R}^{4m}$ as follows:

Let $F(y)(v) = F_1(v)$ and $G(y)(v) = G_1(v)$ for $v \in E(\eta)$ and let

$$F(y)(e_{2m}) = e_{4m}, \quad F(y)(e_{4m}) = -e_{2m}, \quad F(y)(y) = iy, \quad F(y)(iy) = -y, \quad G(y)(e_{2m}) = y, \quad G(y)(e_{4m}) = -iy, \quad G(y)(y) = -e_{2m}, \quad G(y)(iy) = e_{4m}.$$  

Here $e_i$'s denote the standard basis elements of $\mathbb{R}^{4m}$. Clearly, $F(y)^2 = -I$, $G(y)^2 = -I$, $F(y)G(y) = -G(y)F(y)$ for each $y \in \mathbb{R}^{4m}$, where $I$ is the identity map of $\mathbb{R}^{4m}$.

$F$ and $G$ will be used in the construction of a cross section of the fibration

$$O(4m - 1) \rightarrow O(4m) \rightarrow S^{4m-1}.$$  

Consider $S^{4m-3}$ as the set of elements of $S^{4m-1}$ which has 0 in $2m$th entry and $4m$th entry. Then each element $x \in S^{4m-1}$ can be written uniquely of the form

$$x = \alpha e_{2m} + \beta e_{4m} + \gamma y, \quad y \in S^{4m-3}, \gamma \geq 0, \alpha^2 + \beta^2 + \gamma^2 = 1.$$  

Now for each $x \in S^{4m-1}$ let $\sigma(x) = \alpha I + \beta F(y) + \gamma G(y)$. Then we have

$$\sigma(x)\sigma(x)^t = (\alpha I + \beta F(y) + \gamma G(y))(\alpha I - \beta F(y) - \gamma G(y)) = \alpha^2 I - \beta^2 F(y)^2 - \gamma^2 G(y)^2 = (\alpha^2 + \beta^2 + \gamma^2)I = I$$

proving that $\sigma(x) \in O(4m)$. Also, we have

$$\sigma(x)(e_{2m}) = \alpha I(e_{2m}) + \beta F(y)(e_{2m}) + \gamma G(y)(e_{2m}) = \alpha e_{2m} + \beta e_{4m} + \gamma y = x,$$

which shows that $\sigma(x)$ is a cross section. Thus $S^{4m-1}$ is parallelisable, which is a contradiction as we have $m > 2$. This completes the proof of the theorem.

References


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