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ALMOST-QUATERNION $(m-1)$-SUBSTRUCTURES ON S^{4m-3}

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ABSTRACT. We prove that S^{4m-3} admits an almost-quaternion $(m-1)$-substructure if and only if $m=2$, completing the missing case in our paper *On quaternionic James numbers and almost-quaternion substructures on the sphere*.

In [2], an almost-quaternion substructure on an orientable n-manifold M is defined as the reduction of the structure group of $T(M)$ from $SO(n)$ to $Sp(k)\times SO(n-4k)$. This is equivalent to the existence of a $4k$-dimensional subbundle of $T(M)$ together with two normalized almost complex structure maps F, G (see [1]), defined on the total space of the subbundle such that $FG=-GF$. In [2] we have given a theorem which describes the values of k and n such that S^n admits an almost-quaternion k-substructure. There, all the cases were covered except for the case $n=4m-3$ and $k=m-1$ for some m. In this note we prove the following theorem about this case:[1]

THEOREM. *S^{4m-3} admits an almost-quaternion $(m-1)$-substructure if and only if $m=2$.*

PROOF. First let $m=2$. Since $\pi_4(U(2)/Sp(1))=0$, it follows that the fibration

$$U(2)/Sp(1)\to U(3)/Sp(1)\to S^5$$

admits a cross section. But then the fibration

$$SO(5)/Sp(1)\to SO(6)/Sp(1)\to S^5$$

admits a cross section proving that S^5 admits an almost-quaternion 1-substructure.

Next assume $m>2$ and let us assume that S^{4m-3} admits an almost-quaternion $(m-1)$-substructure. Then there exists a $(4m-4)$-dimensional subbundle ξ of $T(S^{4m-3})$ and normalized almost-complex substructure maps F, G defined on the total space $E(\xi)$ such $FG=-GF$. Also, there exists an orthonormal 1-frame $x\to w(x)$ such that the fiber of $E(\xi)$ at x is orthogonal to $w(x)$.

On the other hand let η be the subbundle of $T(S^{4m-3})$ whose fiber at x is the orthogonal complement of the 1-frame $x\to ix$ where i denotes the usual complex structure on $\mathbf{R}^{4m-2}$. On the total space $E(\eta)$ we will define two normalized almost complex substructures F_1, G_1 such that $F_1G_1=-G_1F_1$.

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[1] I have just been informed by Hideaki Ōshima that he had obtained another proof of the same result using homotopy exact sequences of certain fibrations.

Let $R(x)$ be the identity if $ix = w(x)$. Assume $ix \neq w(x)$. Let $\beta(x) = w(x) - (w(x) \cdot ix)ix$. Then $\{ix, \beta(x)/\|\beta(x)\|\}$ is an orthonormal basis for the subspace spanned by ix and $w(x)$. Let $R(x)$ be the orthogonal transformation taking ix to $w(x)$, $\beta(x)/\|\beta(x)\|$ to $-(w(x) \cdot \beta(x))ix/\|\beta(x)\| + (w(x) \cdot ix)\beta(x)/\|\beta(x)\|$, and leaving the orthogonal complement of the subspace spanned by ix and $w(x)$ invariant. Note that the restriction of $R(x)$ to the subspace spanned by ix and $w(x)$ is a rotation by the angle between ix and $w(x)$.

Now, on the total space $E(\eta)$ we define normalized almost complex substructures F_1, G_1 by

$$F_1(v) = R(x)^{-1}(F(R(x)v)), \qquad G_1(v) = R(x)^{-1}(G(R(x)v))$$

for all $x \in S^n$ and $v \in T_xS^n \cap E(\eta)$. Then clearly $F_1G_1 = -G_1F_1$.

We will show that for $m > 2$ this leads to a contradiction. To do this we slightly change the proof of Kirchoff's theorem (Theorem 41.19 of [**3**]) as follows:

For each $y \in S^{4m-3}$ we define linear transformations

$$\tilde{F}(y)\colon \mathbf{R}^{4m} \to \mathbf{R}^{4m} \quad \text{and} \quad \tilde{G}(y)\colon \mathbf{R}^{4m} \to \mathbf{R}^{4m}$$

as follows:

Let $\tilde{F}(y)(v) = F_1(v)$ and $\tilde{G}(y)(v) = G_1(v)$ for $v \in E(\eta)$ and let

$$\tilde{F}(y)(e_{2m}) = e_{4m}, \quad \tilde{F}(y)(e_{4m}) = -e_{2m}, \quad \tilde{F}(y)(y) = iy, \quad \tilde{F}(y)(iy) = -y,$$

$$\tilde{G}(y)(e_{2m}) = y, \quad \tilde{G}(y)(e_{4m}) = -iy, \quad \tilde{G}(y)(y) = -e_{2m}, \quad \tilde{G}(y)(iy) = e_{4m}.$$

Here e_j's denote the standard basis elements of $\mathbf{R}^{4m}$. Clearly, $\tilde{F}(y)^2 = -I$, $\tilde{G}(y)^2 = -I$, $\tilde{F}(y)\tilde{G}(y) = -\tilde{G}(y)\tilde{F}(y)$ for each $y \in \mathbf{R}^{4m}$, where I is the identity map of $\mathbf{R}^{4m}$.

$\tilde{F}$ and $\tilde{G}$ will be used in the construction of a cross section of the fibration

$$O(4m-1) \to O(4m) \to S^{4m-1}.$$

Consider S^{4m-3} as the set of elements of S^{4m-1} which has 0 in $2m$th entry and $4m$th entry. Then each element $x \in S^{4m-1}$ can be written uniquely of the form

$$x = \alpha e_{2m} + \beta e_{4m} + \gamma y, \qquad y \in S^{4m-3}, \gamma \geqslant 0, \alpha^2 + \beta^2 + \gamma^2 = 1.$$

Now for each $x \in S^{4m-1}$ let $\sigma(x) = \alpha I + \beta\tilde{F}(y) + \gamma\tilde{G}(y)$. Then we have $\sigma(x)\sigma(x)^t = [\alpha I + \beta\tilde{F}(y) + \gamma\tilde{G}(y)]\,[\alpha I - \beta\tilde{F}(y) - \gamma\tilde{G}(y)] = \alpha^2 I - \beta^2\tilde{F}(y)^2 - \gamma^2\tilde{G}(y)^2 = (\alpha^2 + \beta^2 + \gamma^2)I = I$ proving that $\sigma(x) \in O(4m)$. Also, we have

$$\sigma(x)(e_{2m}) = \alpha I(e_{2m}) + \beta\tilde{F}(y)(e_{2m}) + \gamma\tilde{G}(y)(e_{2m}) = \alpha e_{2m} + \beta e_{4m} + \gamma y = x,$$

which shows that $\sigma(x)$ is a cross section. Thus S^{4m-1} is parallelisable, which is a contradiction as we have $m > 2$. This completes the proof of the theorem.

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