

RESOLVING ACYCLIC IMAGES OF NONORIENTABLE THREE-MANIFOLDS

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ABSTRACT. We show that every 1-LC \mathbf{Z}_2 -homology 3-manifold (without boundary) which is an almost 1-acyclic (over \mathbf{Z}_2) proper image of a *nonorientable* 3-manifold M (without boundary) is a resolvable generalized 3-manifold. The analogous result for the case when M is *orientable* was recently proved by J. L. Bryant and R. C. Lacher.

1. Introduction. A space X is said to be *locally simply connected* (1-LC) if for every $x \in X$ and every neighborhood $U \subset X$ of x there is a neighborhood $V \subset U$ of x such that any loop in V is null-homotopic in U . A compact subset Y of an ANR X is *cell-like* if for every neighborhood $U \subset X$ of Y there is a neighborhood $V \subset U$ of Y such that V is contractible in U . A mapping f of an ANR M onto a space N is *cell-like* (resp. *monotone*) if for every $x \in N$, $f^{-1}(x)$ is a cell-like set (resp. compact and connected). A mapping $f: X \rightarrow Y$ is *proper* if it is closed and if $f^{-1}(y)$ is compact for all $y \in Y$.

Let R be a principal ideal domain. A metrizable space X is an *R -homology n -manifold* (with respect to singular homology and without boundary) provided $H_*(X, X - \{x\}; R) \cong H_*(\mathbf{R}^n, \mathbf{R}^n - \{0\}; R)$ for each $x \in X$, where $H_*(; R)$ is the singular homology with coefficients in R . A *generalized n -manifold* is a euclidean neighborhood retract (ENR) that is also a \mathbf{Z} -homology n -manifold. An *n -dimensional resolution* of a space X is a pair (M, f) where M is an n -manifold without boundary and $f: M \rightarrow X$ is a proper, cell-like onto mapping.

J. L. Bryant and R. C. Lacher [2, Theorem 2] have proved that every locally contractible 1-acyclic over \mathbf{Z}_2 image X of a 3-manifold M without boundary admits a resolution. In particular, X is a generalized 3-manifold. A refinement of their proof enabled them to omit the acyclicity hypothesis over a 0-dimensional set, provided that M was orientable [2, Theorem 3]. We prove that orientability is not necessary.

THEOREM 1.1. *Let f be a closed, monotone mapping from a 3-manifold M without boundary onto a locally simply connected \mathbf{Z}_2 -homology 3-manifold X . Suppose there is a 0-dimensional set $Z \subset X$ such that $\check{H}^1(f^{-1}(x); \mathbf{Z}_2) = 0$ for all $x \in X - Z$. Then the*

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set $C = \{x \in X \mid f^{-1}(x) \text{ is not cell-like}\}$ is locally finite in X . Moreover, X is a resolvable generalized 3-manifold.

As a corollary we obtain a partial converse in dimension 3 to the well-known fact that a cell-like upper semicontinuous decomposition G of an n -manifold M without boundary always yields a generalized n -manifold (if $n \geq 4$ one must assume, in addition, that M/G is finite dimensional) [5, 7].

COROLLARY 1.2. *Let G be a 0-dimensional upper semicontinuous decomposition of a closed 3-manifold M such that M/G is a 1-LC \mathbf{Z}_2 -homology 3-manifold. Then the set $C = \{g \in G \mid g \text{ is not cell-like}\}$ is finite.*

REMARK 1.3. Let $\pi: M \rightarrow M/G$ denote the quotient map, H_G the collection of all nondegenerate elements of G , and N_G their union.

(1) The Hopf maps or the Bing map [1] show that if $\pi(N_G)$ is a 1-manifold then all nondegenerate elements of G may fail to be cell-like.

(2) Spine maps [1] show that $C = \{g \in G \mid g \text{ is not cell-like}\}$ may have any finite number of elements even when $C = H_G$.

(3) An easy modification of the construction of the Whitehead continuum [12] shows that all nondegenerate elements of G may fail to be cellular even when $\pi(N_G)$ is a Cantor set and G is cell-like. (For details see [11].)

2. Neighborhoods of compacta in nonorientable 3-manifolds. Under some additional hypotheses, Knoblauch’s finiteness theorem [4, Theorem 1] extends to non-orientable 3-manifolds.

PROPOSITION 2.1. *For every closed nonorientable 3-manifold M there exists an integer K such that if $X_1, \dots, X_{K+1} \subset M$ are pairwise disjoint compact sets and each X_i has a neighborhood $U_i \subset M$ such that the inclusion-induced homomorphism $H_1(U_i - X_i; \mathbf{Z}_2) \rightarrow H_1(M; \mathbf{Z}_2)$ is trivial, then at least one X_i has a neighborhood in M which embeds in \mathbf{R}^3 .*

PROOF. We shall suppress the \mathbf{Z}_2 coefficients from the notation. Let $X_1, \dots, X_n \subset M$ be pairwise disjoint compact sets and suppose each X_i has a neighborhood $U_i \subset M$ such that the inclusion-induced homomorphism $H_1(U_i - X_i) \rightarrow H_1(M)$ is trivial, and if $i \neq j$ then $U_i \cap U_j = \emptyset$. Let $X = \bigcup_{i=1}^n X_i$ and $U = \bigcup_{i=1}^n U_i$. Consider the following commutative diagram:

$$\begin{array}{ccccccc}
 \bigoplus_{i=1}^n H_1(U_i - X_i) & \rightarrow & \bigoplus_{i=1}^n H_1(U_i) & & & & \\
 \downarrow \cong & & \downarrow \cong & & & & \\
 \cdots \rightarrow H_1(U - X) & \rightarrow & H_1(U) & \rightarrow & H_1(U, U - X) & \rightarrow \cdots & \\
 & & \downarrow & & \Psi \downarrow \cong & & \\
 \cdots \rightarrow H_1(M - X) & \rightarrow & H_1(M) & \rightarrow & H_1(M, M - X) & \rightarrow \cdots &
 \end{array}$$

where the horizontal sequences are exact and Ψ is the excision isomorphism. It is easy to see that the image of the inclusion-induced homomorphism $H_1(U) \rightarrow H_1(M)$

is the direct sum of the images of the inclusion-induced homomorphisms $H_1(U_i) \rightarrow H_1(M)$, $1 \leq i \leq n$. So if we let $\beta_1 = \text{rank } H_1(M)$ then $n - \beta_1$ of the homomorphisms $H_1(U_i) \rightarrow H_1(M)$ are trivial. It follows by [8, Lemma (4.1)] that $n - \beta_1$ of the neighborhoods U_i are orientable. Let $k(\tilde{M})$ be the Knoblauch number of the orientable 3-manifold double cover of M [4, Theorem 1]. Since every orientable neighborhood lifts in \tilde{M} to two (homeomorphic) copies, it follows that if $2(n - \beta_1) > k(\tilde{M})$ then some \tilde{X}_i has a neighborhood in M which embeds in \mathbf{R}^3 . We can now determine the number K from the equation $2(K - \beta_1) = k(\tilde{M})$.

PROPOSITION 2.2. *Let K be a compact connected subset of the interior of a 3-manifold M . Suppose K does not separate its connected neighborhoods and, for every neighborhood $U \subset M$ of K there exists a neighborhood $V \subset U$ of K such that the inclusion-induced homomorphism $H_1(V - K; \mathbf{Z}_2) \rightarrow H_1(U; \mathbf{Z}_2)$ is trivial. Then $K = \bigcap_{i=1}^\infty N_i$ where each $N_i \subset \text{int } M$ is a compact 3-manifold with boundary satisfying the following properties:*

- (i) $N_{i+1} \subset \text{int } N_i$;
- (ii) N_i is obtained from a compact 3-manifold Q_i with a 2-sphere boundary by adding to ∂Q_i a finite number of orientable (solid) 1-handles;
- (iii) the inclusion-induced homomorphism $H_1(\partial N_{i+1}; \mathbf{Z}_2) \rightarrow H_1(N_i; \mathbf{Z}_2)$ is trivial;
- (iv) there is a homeomorphism $h_i: N_i \rightarrow N_i$ such that $h_i|_{\partial N_i} = \text{identity}$ and $h_i(Q_i^*) = Q_{i+1}$, where $Q_i^* \subset \text{int } Q_i$ is formed by pushing Q_i into $\text{int } Q_i$ along a collar of ∂Q_i .

REMARK 2.3. An examination of the proofs in [10] shows that the orientability hypothesis can be removed from all results in [10] if one uses Proposition 2.2 in place of [9, Theorem 2].

PROOF OF PROPOSITION 2.2. By [13, Theorem 2], $K = \bigcap_{i=1}^\infty N_i$ where each $N_i \subset \text{int } M$ is a compact 3-manifold with boundary satisfying (i) and (ii) above (the orientability of the 1-handles follows by [8, Lemma (4.1)]). By choosing an appropriate subsequence of $\{N_i\}$ we can satisfy (iii). We prove (iv). Let $K_i \subset \text{int } Q_i$ be a spine of Q_i . Let \hat{Q}_i be the closed 3-manifold we obtain by attaching a 3-cell to ∂Q_i . For each $i \geq 1$, $N_i = (N_i/K_i) \# \hat{Q}_i$ (the interior connected sum [3]). Since N_i is nonorientable, it admits a unique normal, prime decomposition $N_i = M_1 \# \dots \# M_n$, $M_i \neq S^2 \times S^1$ [3, Theorem (3.15) and Lemma (3.17)]. Consider normal, prime decompositions of N_i/K_i and \hat{Q}_i ($i \geq 1$). Since N_i/K_i is clearly orientable, its normal, prime decomposition $N_i/K_i = A_1 \# \dots \# A_p \# B_1 \# \dots \# B_q$ may contain $p > 0$ summands $A_i = S^2 \times S^1$. On the other hand, \hat{Q}_i is nonorientable (since N_i is) so its normal prime decomposition $\hat{Q}_i = C_1 \# \dots \# C_r$ contains no $S^2 \times S^1$ summands. By [3, Lemma (3.17)] we may replace each A_i by $P =$ the nonorientable S^2 -bundle over S^1 to get a normal, prime decomposition $N_i = P \# \dots \# P \# B_1 \# \dots \# B_q \# C_1 \# \dots \# C_r$ (p summands P) of N_i . It follows by the uniqueness of normal, prime decompositions that $p + q + r = n$ and that after a suitable permutation of the summands each C_i is homeomorphic to some M_i . We may conclude that among any $n + 1$ \hat{Q}_i 's at least two have the same prime summands (up to a homeomorphism). By choosing an appropriate subsequence of $\{Q_i\}$ we may henceforth assume that for each $i \leq j$ there is a homeomorphism $s_{ij}: Q_i \rightarrow Q_j$.

We first construct h_1 . The identity on ∂N_1 induces a homeomorphism $t'_{ij}: \partial(N_1/K_i) \rightarrow \partial(N_1/K_j)$ for each $i \leq j$. Using Dehn's lemma we can extend t'_{ij} to a homeomorphism $t_{ij}: N_1/K_i \rightarrow N_1/K_j$. Finally, define $h_{ij}: N_1 \rightarrow N_1$ by $h_{ij}(x) = s_{ij}(x)$ if $x \in Q_i$ and $h_{ij}(x) = t_{ij}(x)$ if $x \in N_1 - Q_i$. Clearly, $h_{ij}|_{\partial N_1} = \text{identity}$ and $h_{ij}(Q_i^*) = Q_j^*$. We define h_2 as the composition of h_{12} and a homeomorphism of N_1 that is the identity outside a neighborhood of ∂Q_2 in N_2 and pushes Q_2^* onto Q_2 . We can get $h_i, i \geq 2$, in a similar way. For details see [11].

3. The proof of Theorem (1.1). We shall suppress the \mathbf{Z}_2 coefficients from the notation. Let $A = \{x \in X | \check{H}^1(f^{-1}(x)) \neq 0\}$. By [6, Theorem (4.1)], A is locally finite in X . Let $B = \{x \in X | f^{-1}(x) \text{ has no neighborhood embeddable in } \mathbf{R}^3\}$. In order to show that B is locally finite in X it suffices, by Proposition 2.1, to prove that for each $x \in X, f^{-1}(x)$ possess a neighborhood $U \subset M$ such that

$$H_1(U - f^{-1}(x)) \rightarrow H_1(M)$$

is trivial. So let $x \in X$. Since A is locally finite there is a neighborhood $W \subset X$ of x such that $W \cap A \subset \{x\}$. By hypothesis X is LC^1 so there is a connected neighborhood $W' \subset W$ of x such that any loop in W' is null-homotopic in W . Consider the following commutative diagram:

$$\begin{array}{ccc} H_1(f^{-1}(W') - f^{-1}(x)) & \xrightarrow{i'_*} & H_1(f^{-1}(W) - f^{-1}(x)) \\ \cong \downarrow f|_* & & \cong \downarrow f|_* \\ H_1(W' - \{x\}) & \rightarrow & H_1(W - \{x\}) \\ \downarrow j'_* & & \downarrow j_* \\ H_1(W') & \xrightarrow{i_*} & H_1(W) \end{array}$$

where the horizontal homomorphisms are induced by inclusions, $f|_*$ is the Vietoris-mapping theorem isomorphism [7, 3.4], while j_* and j'_* are the isomorphisms from the homology sequence of the pairs $(W, W - \{x\})$ and $(W', W' - \{x\})$, respectively. By hypothesis, $i_* = 0$, hence $i'_* = 0$. Thus we may apply Proposition 2.1. By Proposition 2.2, $f^{-1}(x)$ is definable by (orientable) cubes with handles for all $x \in X - B$, so by [9, Theorem 3], $f^{-1}(x)$ has the 1-UV property. Since cubes with handles have no higher homotopy, each $f^{-1}(x)$ has the UV^∞ property and hence $C \subset B$ (cf. [7]). Therefore, C is locally finite in X . In particular, $X - C$ is finite dimensional by [5]. A resolution of X is now obtained by improving f over the points of C . This is done similarly as in [2]. For details see [11].

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