ORIENTABLE PRODUCTS IN $\mathcal{N}$

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Abstract. It is well known that the square $W \times W$ of any smooth closed manifold $W$ is cobordant to an orientable manifold. This note shows more specifically that a product $U \times V$ of smooth closed manifolds $U$ and $V$ is cobordant to an orientable manifold if and only if there is a smooth closed manifold $W$ such that $U$ and $V$ are both products of $W$ by orientable manifolds.

The following result will be established:

Theorem. Let $U$ and $V$ be any smooth closed manifolds. Then the product $U \times V$ is cobordant to an orientable manifold if and only if there are orientable manifolds $U_0$ and $V_0$ and a manifold $W$, not necessarily orientable, such that $U$ and $V$ are cobordant to $U_0 \times W$ and $V_0 \times W$, respectively.

One can reformulate the preceding theorem as follows. Let $\mathcal{N}$ be the unoriented cobordism ring, and let $\mathcal{N}_0 \subset \mathcal{N}$ be the subring whose classes can be represented by smooth closed orientable manifolds; that is, $\mathcal{N}_0$ is the image of the oriented cobordism ring $\Omega$ under the forgetful homomorphism $\Omega \to \mathcal{N}$. One wants to show that $u \in \mathcal{N}$ and $v \in \mathcal{N}$ satisfy $uv \in \mathcal{N}_0$ if and only if there are elements $u_0 \in \mathcal{N}_0$, $v_0 \in \mathcal{N}_0$ and $w \in \mathcal{N}$ such that $u = u_0w$ and $v = v_0w$.

Here are the two steps of the proof. One first introduces the subring $\mathcal{W} \subset \mathcal{N}$ of Wall [4], and one uses the fact that both $\mathcal{W}$ and $\mathcal{N}$ are unique factorization domains to prove the preceding assertion with $\mathcal{W}$ substituted for $\mathcal{N}_0$. One then uses Wall's characterization of $\mathcal{N}_0 \subset \mathcal{W}$ as the kernel of a derivation $\mathcal{W} \to \mathcal{W}$ to reinstate $\mathcal{N}_0$ in the result. The author thanks Professor Robert E. Stong for his contribution to the proof.

Recall from [3] that $\mathcal{N}$ is a $\mathbb{Z}/2$ polynomial algebra $\mathbb{Z}/2[x_n \mid n \neq 2^r - 1]$ with one generator $x_n \in \mathcal{N}$ in each degree $n \geq 0$ not of the form $2^r - 1$ for $r > 0$. In particular, $x_0 \in \mathcal{N}$ is the identity element, consisting of the class $\{\ast\}$ of the singleton manifold $\{\ast\}$; the generators $x_n \in \mathcal{N}$ for $n \geq 4$ are not uniquely defined, of course, since one can always replace $x_n$ by $x_n + (\text{decomposable elements})$ whenever $n \geq 4$. For notational convenience we write $x(2^s)$ in place of $x_{2^s}$ to denote any generator of degree $2^s$.

In [4] Wall chooses specific generators $x_n$ of $\mathcal{N}$, $n \neq 2^r - 1$, and he introduces the $\mathbb{Z}/2$ polynomial subalgebra $\mathbb{Z}/2[x_n, x(2^s)^2 \mid n \neq 2^r - 1, n \neq 2^s]$ as a subring $\mathcal{W} \subset \mathcal{N}$. It is known that $\mathcal{W}$ consists of all classes in $\mathcal{N}$ represented by at least one manifold with an integral first Stiefel-Whitney class, and since $\mathcal{N}_0$ consists of all classes in $\mathcal{N}$ represented by at least one manifold with a vanishing first Stiefel-Whitney class, the inclusion $\mathcal{N}_0 \subset \mathcal{W}$ is clear. Since $\mathcal{W}$ and $\mathcal{N}$ are polynomial rings over a field $\mathbb{Z}/2$
they are both unique factorization domains; unique factorization is not available in \( \mathcal{N}_0 \) itself.

Here is the classical property of squares in \( \mathcal{N} \), quoted earlier. The even-degree generators \( x_{2k} \) appearing in the following proof differ by decomposable elements from the even-degree generators used to describe \( \mathcal{W} \); however, this substitution does not affect the result.

**Lemma 1.** For any \( w \in \mathcal{N} \) one has \( w^2 \in \mathcal{N}_0 \); a fortiori \( w^2 \in \mathcal{W} \).

**Proof.** It suffices to select generators \( x_n \) of \( \mathcal{N} \) such that each square \( x_n^2 \) is represented by an orientable manifold; in fact, the classical choices of \([3 \text{ and 1}]\) have this property. Specifically, if \( n \) is an even number \( 2k \geq 0 \) one can choose the cobordism class \( [RP^{2k}] \) of the real projective space \( RP^{2k} \) to be a generator \( x_{2k} \), as in \([3]\), where \( RP^0 = \{*\} \); however, according to \([4]\) the square \( RP^{2k} \times RP^{2k} \) is cobordant to the (canonically oriented) complex projective space \( CP^{2k} \). For any odd \( n \neq 2' - 1 \) one can choose the cobordism class \( [P(n)] \) of the Dold manifold \( P(n) \) to be a generator \( x_n \), as in \([1]\); since each \( P(n) \) is itself orientable the same is true of \( P(n) \times P(n) \).

(In fact, since the squares \( x_n^2 \) of the preceding odd-dimensional generators \( x_n \) all lie in a proper subring of \( \mathcal{N}_0 \), the preceding proof shows that all squares \( w^2 \) lie in some fixed proper subring of \( \mathcal{N}_0 \); see \([2]\), for example, for further details.)

**Lemma 2.** Let \( u \) and \( v \) be any nonzero elements of \( \mathcal{N} \) such that \( uv = v_0 \in \mathcal{W} \), where \( v_0 \) is necessarily nonzero. Then there is a nonzero \( u_0 \in \mathcal{W} \) such that \( v_0 u = u_0 v \).

**Proof.** If \( u_0 = u^2 \in \mathcal{W} \) then \( v_0 u = uvu = u^2 v = u_0 v \).

One can regard \( \mathcal{N} \) as an algebra generated over \( \mathcal{W} \) by the identity element \( x_0 \in \mathcal{N} \) and the specific generators \( x(2^s) \in \mathcal{N} \) of \([4]\); a fortiori one can regard \( \mathcal{N} \) as a \( \mathcal{W} \)-module spanned by \( x_0 \) and all products \( x(2^{s_1}) \cdots x(2^{s_q}) \) such that \( s_1 < \cdots < s_q \).

**Lemma 3.** \( \mathcal{N} \) is the free \( \mathcal{W} \)-module spanned by \( x_0 \) and all products \( x(2^{s_1}) \cdots x(2^{s_q}) \) such that \( s_1 < \cdots < s_q \).

**Proof.** Clearly there is a direct sum decomposition \( \mathcal{N} = \mathcal{W} \oplus \mathcal{M} \) over \( \mathcal{W} \), where the summand \( \mathcal{W} \) consists of \( \mathcal{W} \)-multiples of \( x_0 \) and the summand \( \mathcal{M} \) is spanned by the remaining generators. Given any \( \mathcal{W} \)-linear relation \( \sum u(s_1, \ldots, s_q)x(2^{s_1}) \cdots x(2^{s_q}) = 0 \in \mathcal{M} \), one multiplies by any fixed \( x(2^{s_1}) \cdots x(2^{s_q}) \) to obtain a \( \mathcal{W} \)-linear relation in \( \mathcal{W} \oplus \mathcal{M} \) for which the coefficient of \( x_0 \) is \( u(s_1, \ldots, s_q)x(2^{s_1})^2 \cdots x(2^{s_q})^2 = 0 \in \mathcal{W} \). Since \( \mathcal{W} \) is a unique factorization domain it follows that all coefficients \( u(s_1, \ldots, s_q) \in \mathcal{W} \) in the given \( \mathcal{W} \)-linear relation vanish, hence that the generators \( \{x(2^{s_1}) \cdots x(2^{s_q}) \mid s_1 < \cdots < s_q\} \) of the summand \( \mathcal{M} \) are linearly independent.

**Lemma 4.** Let \( u \) and \( v \) be any nonzero homogeneous elements of \( \mathcal{N} \) such that \( uv \in \mathcal{W} \); then there are homogeneous elements \( u_0 \in \mathcal{W} \) and \( v_0 \in \mathcal{W} \) and \( w \in \mathcal{N} \) such that \( u = u_0 w \) and \( v = v_0 w \).

**Proof.** One forms the quotient field \( \mathcal{F} \) of the unique factorization domain \( \mathcal{W} \); the tensor product \( \mathcal{F} \otimes \mathcal{N} \) over \( \mathcal{W} \) is then a vector space over \( \mathcal{F} \). According to Lemma 2 the nonzero elements \( 1 \otimes u \in \mathcal{F} \otimes \mathcal{N} \) and \( 1 \otimes v \in \mathcal{F} \otimes \mathcal{N} \) lie in the same one-dimensional subspace of \( \mathcal{F} \otimes \mathcal{N} \), and Lemma 3 provides a unique basis element of the form \( 1 \otimes w \) with \( w \in \mathcal{N} \) of least degree; trivially, \( \deg w \leq \deg u \) and \( \deg w \leq
deg \nu. There is then an element \( u_0/w_0 \in \mathcal{F} \) such that \( 1 \otimes u = u_0/w_0 \otimes w \), where one may as well assume that \( u_0 \) and \( w_0 \) have no factors in common. One squares this relation and applies Lemma 1 to conclude that
\[
\nu^2 \otimes x_0 = u_0^2 w_0^2 / w_0^2 \otimes x_0 \in \mathcal{F} \otimes \mathcal{N},
\]
hence \( u_0^2 \nu^2 = u_0^2 w_0^2 \in \mathcal{W} \) by Lemma 3. Since \( u_0 \) and \( w_0 \) have no factors in common it follows that \( u_0^2 \) divides \( \nu^2 \) in the unique factorization domain \( \mathcal{W} \), hence \( w_0 \) divides \( w \) in the unique factorization domain \( \mathcal{N} \). However, \( w_0 \) itself belongs to \( \mathcal{W} \), so it cannot be of positive degree without contradicting the minimality of \( \deg \nu \); hence \( u_0 \) is the identity element \( x_0 \), so \( u = u_0 w \) for a unique \( u_0 \in \mathcal{W} \). Similarly \( v = v_0 w \) for a unique \( v_0 \in \mathcal{W} \), which completes the proof.

The goal of the next few lemmas is to replace \( \mathcal{W} \) by \( \mathcal{N} \) in Lemma 4.

**Lemma 5.** Let \( p_1, \ldots, p_s \) be distinct primes in the unique factorization domain \( \mathcal{N} \), and let \( r_1, \ldots, r_s \) be natural numbers such that \( p_1^{r_1} \cdots p_s^{r_s} \in \mathcal{W} \); then for each \( i = 1, \ldots, s \) either \( p_i \in \mathcal{W} \) or \( r_i \) is even.

**Proof.** Set \( u = p_1^{r_1} \) and \( v = p_2^{r_2} \cdots p_s^{r_s} \) and observe that \( u \) and \( v \) have no common factors; hence \( p_1^{r_1} = u = u_0 \in \mathcal{W} \) by Lemma 4. If \( r_1 \) is an odd number \( 2q + 1 \) then set \( u = p_1 \) and \( v = p_2^{2q} \) and use Lemma 4 once again to conclude either that \( p_1 \in \mathcal{W} \) or that \( p_2^{2q-1} \in \mathcal{W} \); if \( p_1 \) were not in \( \mathcal{W} \) one could then set \( u = p_1 \) and \( v = p_2^{2q-2} \) in Lemma 4 to conclude either that \( p_1 \in \mathcal{W} \) or that \( p_1^{2q-3} \in \mathcal{W} \), and so forth until one could not escape the conclusion \( p_1 \in \mathcal{W} \). A similar argument applies to each of the remaining prime powers \( p_2^{r_2}, \ldots, p_s^{r_s} \).

For later reference we remark that Lemma 5 is in fact equivalent to Lemma 4; the proof of the converse assertion appears in disguise in Lemma 9.

**Lemma 6.** If \( u \) and \( v \) are nonzero elements of \( \mathcal{N} \) such that \( u \in \mathcal{N}_0 \) and \( uv \in \mathcal{N}_0 \), then \( v \in \mathcal{N}_0 \).

**Proof.** See p. 308 of [4].

**Lemma 7.** Let \( p_1, \ldots, p_s \) be distinct primes in the unique factorization domain \( \mathcal{N} \) such that \( p_1 \cdots p_s \in \mathcal{N}_0 \); then each of \( p_1, \ldots, p_s \) belongs to \( \mathcal{N}_0 \).

**Proof.** Since \( \mathcal{N}_0 \subset \mathcal{W} \) Lemma 5 implies that each of \( p_1, \ldots, p_s \) belongs to \( \mathcal{W} \). If \( p_1 \cdots p_{s-1} \in \mathcal{N}_0 \) then Lemma 6 implies \( p_s \in \mathcal{N}_0 \). Hence to prove that \( p_s \in \mathcal{N}_0 \), for example, it remains only to consider the case that \( p_1 \cdots p_{s-1} \notin \mathcal{N}_0 \), where \( p_1 \cdots p_{s-1} \in \mathcal{W} \). Wall [4] provides an exact sequence \( 0 \rightarrow \mathcal{W} \rightarrow \mathcal{N} \rightarrow 0 \) in which \( r \) is the forgetful homomorphism from \( \Omega \) onto \( \mathcal{N}_0 \subset \mathcal{W} \) and \( \partial \) is a derivation of degree \(-1\) over \( \mathbb{Z}/2 \); thus \( \mathcal{N}_0 = \ker \partial \). (Wall uses the notation \( \partial_1 \) for \( \partial \); the preceding exact sequence is used later in Wall's paper to construct the portion \( \Omega \rightarrow \mathcal{W} \rightarrow \partial \) of the exact triangle which determines \( \Omega \).) In any event, if \( p_1 \cdots p_{s-1} \in \mathcal{W} \) and \((p_1 \cdots p_{s-1})p_s \in \mathcal{W} \) satisfy \( p_1 \cdots p_{s-1} \notin \mathcal{N}_0 \) and \((p_1 \cdots p_{s-1})p_s \in \mathcal{N}_0 \), one has \( \partial(p_1 \cdots p_{s-1}) \neq 0 \in \mathcal{W} \) and \( \partial((p_1 \cdots p_{s-1})p_s) = 0 \in \mathcal{W} \) for the derivation \( \partial \). Hence
\[
(\partial(p_1 \cdots p_{s-1}))p_s + (p_1 \cdots p_{s-1})\partial p_s = \partial(p_1 \cdots p_s) = 0
\]
for \( \partial(p_1 \cdots p_{s-1}) \neq 0 \), and, since \( p_s \) does not divide \( p_1 \cdots p_{s-1} \), it follows that \( p_s \) divides \( \partial p_s \). Since \( \deg(\partial p_s) = (\deg p_s) - 1 \), this implies \( \partial p_s = 0 \); that is, \( p_s \in \ker \partial = \mathcal{N}_0 \) as claimed.

The next lemma is an analog of Lemma 5.

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**Lemma 8.** Let $p_1, \ldots, p_s$ be distinct primes in the unique factorization domain $\mathcal{N}$, and let $r_1, \ldots, r_s$ be natural numbers such that $p_1^{r_1} \cdots p_s^{r_s} \in \mathcal{N}_0$; then for each $i = 1, \ldots, s$ either $p_i \in \mathcal{N}_0$ or $r_i$ is even.

**Proof.** Write $p_1^{r_1} \cdots p_s^{r_s} = uv$, where $u$ contains all squares and $v$ is of the form $p_1^{q_1} \cdots p_s^{q_s}$, where each of $q_1, \ldots, q_s$ is either 0 or 1. In case $q_i = 0$ the corresponding $r_i$ is even, and $v$ is the product of the remaining distinct primes among $p_1, \ldots, p_s$. Since $u \in \mathcal{N}_0$ by Lemma 1, it follows from Lemma 6 that $v \in \mathcal{N}_0$; hence, in case $q_i = 1$, Lemma 7 implies $p_i \in \mathcal{N}_0$.

**Lemma 9.** Let $u$ and $v$ be any nonzero homogeneous elements of $\mathcal{N}$ such that $uv \in \mathcal{N}_0$; then there are homogeneous elements $u_0 \in \mathcal{N}_0$, $v_0 \in \mathcal{N}_0$ and $w \in \mathcal{N}$ such that $u = u_0 w$ and $v = v_0 w$.

**Proof.** Let $p_1, \ldots, p_s$ be the distinct primes occurring in the unique factorizations of $u$ and $v$, so that $u = p_1^{m_1} \cdots p_s^{m_s}$ and $v = p_1^{n_1} \cdots p_s^{n_s}$ for natural numbers $m_1, \ldots, m_s, n_1, \ldots, n_s$. The hypothesis $uv \in \mathcal{N}_0$ and Lemma 8 guarantee for each $i = 1, \ldots, s$ either that $p_i \in \mathcal{N}_0$ or that $m_i + n_i$ is even. In the latter case $|m_i - n_i|$ is also even, and it remains only to recall from Lemma 1 that $p_i^2 \in \mathcal{N}_0$ for any $p_i \in \mathcal{N}$.

This completes the proof of the Theorem itself since Lemma 9 is precisely the reformulated version described earlier.

We remark that one can substitute $W$ for $\mathcal{N}_0$ in the proof of Lemma 9 to complete the proof that Lemmas 4 and 5 are equivalent, as promised earlier. Lemmas 8 and 9 are similarly equivalent.

**References**


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