PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 90, Number 1, January 1984

SHORTER NOTES

The purpose of this department is to publish very short papers of unusually elegant and polished character, for which there is no other outlet.

AN ELEMENTARY PROOF OF THE RUSSO-DYE THEOREM

L. TERRELL GARDNER

Those familiar with the useful theorem proved below, first proved in [**1**], may appreciate the simplicity of this proof.

THEOREM. *Let A be a C^*-algebra with unit element* 1 *and unitary group U. Then the closed unit ball in A is the closed convex hull of U.*

PROOF. It suffices to show that if $x \in A_1$, the open unit ball of A, then $x \in \overline{\mathrm{co}}(U)$, the closed convex hull of U. It is easy to reduce this to showing that for every $u \in U$, $y = (x + u)/2 \in \overline{\mathrm{co}}(U)$: For then $U \subset 2\,\overline{\mathrm{co}}(U) - x$, which is closed and convex, so $\overline{\mathrm{co}}(U) \subset 2\,\overline{\mathrm{co}}(U) - x$, or $(x + \overline{\mathrm{co}}(U))/2 \subset \overline{\mathrm{co}}(U)$; if $x_0 \in U$, and $x_{n+1} = (x + x_n)/2$, $x_n \in \overline{\mathrm{co}}(U)$ and $x_n \to x$.

But note that $y = ((xu^{-1} + 1)/2)u$, so (since $\|xu^{-1}\| = \|x\| < 1$) $\|y\| < 1$, and y is invertible. Thus $y = v|y|$, with v unitary and $(y^*y)^{1/2} = |y| = (w + w^*)/2$, where $w = |y| + i(1 - |y|^2)^{1/2}$ is also unitary. This concludes the proof and indeed proves the stronger result: $A_1 - U \subset U + U$.

NOTE ADDED IN PROOF. C. K. Fong has pointed out that essentially the same argument proves also that $A_1 \subset \mathrm{co}(U)$: Extend slightly the segment from u through x to $x' \in A_1$, and apply the given argument to x' instead of x; then $x'_n \in \mathrm{co}(U)$, and for large n, x lies between x'_n and u.

REFERENCES

1. B. Russo and H. A. Dye, *A note on unitary operators in C^*-algebras*, Duke Math. J. **33** (1966), 413–416.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO, TORONTO, ONTARIO, CANADA M5S 1A1

Received by the editors January 18, 1983 and, in revised form, March 22, 1983.
1980 *Mathematics Subject Classification*. Primary 46L05, 47D20.