

SOLVABLE GROUPS WITH π -ISOLATORS

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ABSTRACT. Let π be any nonempty set of prime numbers. A natural number is a π -number precisely if all of its prime factors are in π . A group G is said to have the π -isolator property if for every subgroup H of G , the set $\sqrt[\pi]{H} = \{g \in G; g^n \in H, \text{ for some } \pi\text{-number } n\}$ is a subgroup of G . It is well known that nilpotent groups have the π -isolator property for any nonempty set π of primes. Finitely generated solvable linear groups with finite Prüfer rank, and in particular polycyclic groups, have subgroups of finite index with the π -isolator property if π is the set of all primes. It is shown here that if π is any finite nonempty set of primes and G is a finitely generated solvable group, then G has a subgroup of finite index with the π -isolator property if and only if G is nilpotent-by-finite.

1. Introduction. Let π be a nonempty set of prime numbers. A natural number is a π -number precisely if all of its prime factors are in π . A group G is said to have the π -isolator property if for every subgroup H of G , the set $\sqrt[\pi]{H} = \{g \in G; g^n \in H \text{ for some } \pi\text{-number } n\}$ is a subgroup of G . A subgroup H of G is called π -isolated if $\sqrt[\pi]{H} = H$. The π -isolator of a subgroup H of G is the smallest π -isolated subgroup of G containing H . This is the intersection of all π -isolated subgroups of G containing H . The relation \sim_π on the set of all subgroups of G given by $K \sim_\pi H$ if and only if $\sqrt[\pi]{K} = \sqrt[\pi]{H}$; $K, H \leq G$, is an equivalence relation. Thus if G has the π -isolator property then each of the equivalence classes has a unique maximal member.

If π is the set of all primes, then the π -isolator of H in G is called the *isolator of H in G* and G is said to possess *the isolator property* if G has the π -isolator property where π is the set of all primes. It was shown in [2] that for finitely generated solvable groups those with the isolator property are closely linked to groups with finite (Prüfer) rank. For instance, a torsion-free solvable group of finite rank has a subgroup of finite index with the isolator property. Conversely, if G is a finitely generated nilpotent-by-abelian group with the isolator property, then G has finite rank.

A group G is said to have the *strong isolator property* if it has the isolator property and, in addition, $|\sqrt[\pi]{H} : H|$ is finite for all subgroups H of G . A polycyclic group has a subgroup of finite index with the strong isolator property. Conversely, if G is a finitely generated solvable group with the strong isolator property, then G is polycyclic (see [2, Theorem A]).

Received by the editors November 4, 1982 and, in revised form, April 22, 1983.

1980 *Mathematics Subject Classification*. Primary 20F16; Secondary 20E34.

Key words and phrases. Solvable, nilpotent-by-finite, π -isolators.

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If G is nilpotent (or even locally nilpotent) then it has the π -isolator property for every nonempty set π of primes. On the other hand, if a finitely generated solvable group G has p -isolator property for every prime p , then G is necessarily nilpotent. This follows from three observations. (i) A finite group of this type is nilpotent. (ii) If a finitely generated solvable group is not nilpotent then it has a finite quotient which is not nilpotent (see [3, Theorem 10.51]). (iii) The π -isolator property is quotient closed. In this paper we investigate solvable groups G which have the π -isolator property for some finite (nonempty) set π of primes. It would be conceivable that polycyclic groups would perhaps have this property. The answer is to the contrary and our main result is the following.

THEOREM A. *Let π be a nonempty finite set of primes, and let G be a finitely generated solvable group. Then G has a subgroup of finite index possessing the π -isolator property if and only if G is nilpotent-by-finite.*

2. Proofs. The proof of Theorem A is not direct. We find it desirable to start with the following key lemma. If $\theta \neq 0$ is algebraic over \mathbf{Q} , let $\mathbf{Z}\langle\theta\rangle$ denote the subring of $\mathbf{Q}(\theta)$ generated by $1, \theta, \theta^{-1}$. Define the group Γ_θ as the semidirect product of additive groups $\mathbf{Z}\langle\theta\rangle, \mathbf{Z}$. Thus $\Gamma_\theta = \mathbf{Z}\langle\theta\rangle \rtimes \mathbf{Z}$, and

$$(\alpha, n) \cdot (\beta, m) = (\alpha + \theta^n\beta, n + m); \quad \alpha, \beta \in \mathbf{Z}\langle\theta\rangle, n, m \in \mathbf{Z}.$$

KEY LEMMA. *If π is a finite set of prime numbers so that Γ_θ has the π -isolator property, then either*

- (a) π is empty, or
- (b) θ is a root of unity.

PROOF. Suppose, by way of contradiction, that Γ_θ has the π -isolator property, θ is not a root of unity, and π has a largest element q . For $k \geq 1$ put

$$\lambda_k = 1 + \theta^k + \theta^{2k} + \dots + \theta^{(q-1)k} = \frac{\theta^{kq} - 1}{\theta^k - 1} \neq 0 \text{ in } \mathbf{Z}\langle\theta\rangle.$$

The prime divisors P of $\mathbf{Q}(\theta)$ have corresponding absolute values

$$\|\alpha\|_P = \begin{cases} \text{norm}(P)^{-\text{ord}_P(\alpha)}, & P \text{ non-Archimedean,} \\ |\alpha|, & P \text{ real,} \\ |\alpha|^2, & P \text{ complex,} \end{cases}$$

normalized so that the Product Formula holds:

$$\prod_P \|\alpha\|_P = 1 \quad \text{for } 0 \neq \alpha \in \mathbf{Q}(\theta).$$

Since θ is not a root of unity it has height

$$H = \prod_P \max(1, \|\theta\|_P) > 1$$

(see, for example, Theorem 8 on p. 77 of [4]). Consider the following classes of prime divisors of $\mathbf{Q}(\theta)$:

- S_∞ : Archimedean primes,
- S_θ : $P \notin S_\infty$ so $\text{ord}_P \theta \neq 0$,
- S_π : $P \notin S_\infty \cup S_\theta$ so P lies above some $p \in \pi$.

Then $S = S_\infty \cup S_\theta \cup S_\pi$ is finite and for $\epsilon > 0$ we claim

$$\frac{\|\lambda_k\|_P^{1/k}}{\max(1, \|\theta\|_P)^{q-1}} \begin{cases} = 1 & \text{for } P \notin S \text{ and all } k, \\ \geq 1 - \epsilon & \text{for } P \in S \text{ and infinitely many } k. \end{cases}$$

The lemma is an immediate consequence since the claim shows that

$$\frac{1}{H^{q-1}} = \prod_P \frac{\|\lambda_k\|_P^{1/k}}{\max(1, \|\theta\|_P)^{q-1}} \geq (1 - \epsilon)^{\text{card } S},$$

for infinitely many k , holds for each $\epsilon > 0$. Clearly, we may choose $\epsilon > 0$ to contradict $H > 1$ so it remains only to verify the claim.

Suppose $P \notin S$: since $\|\theta\|_P = 1$ and $\lambda_k \in \mathbf{Z}\langle\theta\rangle$, we need only rule out the possibility that $\text{ord}_P \lambda_k > 0$. If this was the case then λ_k is in the ideal $P_0 = \{\alpha \in \mathbf{Z}\langle\theta\rangle : \text{ord}_P \alpha > 0\}$ of $\mathbf{Z}\langle\theta\rangle$ so (λ_k, kq) is in the subgroup $H = P_0 \rtimes kq\mathbf{Z}$ of Γ_θ . By $(0, -k)^q = (0, -kq)$, $(1, k)^q = (\lambda_k, kq)$ both $(0, -k)$, $(1, k)$ are in $\sqrt[q]{H}$ and since this is a group, by hypothesis, it follows that $(1, 0) = (1, k)(0, -k)$ is in $\sqrt[q]{H}$. This means that $(N, 0) = (1, 0)^N$ is in $H \cap \mathbf{Z}\langle\theta\rangle = P_0$ for some π -number N , hence that $\text{ord}_P N > 0$ contrary to $P \notin S$.

Suppose $P \in S_\theta$: here the claim follows for all $k \geq 1$ from

$$\lambda_k = \frac{1 - \theta^{kq}}{1 - \theta^k} \quad \text{if } \|\theta\|_P < 1 \quad \text{and} \quad \lambda_k = \theta^{k(q-1)} \frac{1 - \theta^{-kq}}{1 - \theta^{-k}} \quad \text{if } \|\theta\|_P > 1.$$

Suppose $P \in S_\pi$: it suffices to show that there is a constant a_P depending only on P (and θ) so that

$$(*) \quad \text{ord}_P(\theta^k - 1) \leq a_P + \text{ord}_P(k) \quad \text{for all } k \geq 1.$$

For then

$$\|\lambda_k\|_P^{1/k} = \left\| \frac{\theta^{kq} - 1}{\theta^k - 1} \right\|_P^{1/k} \geq \|\theta^{kq} - 1\|_P^{1/k} \geq \text{norm}(P)^{-(a_P + \text{ord}_P(kq))/k}$$

which is $> 1 - \epsilon$ for all large enough k .

So let p be the prime number below P , set $e_P = \text{ord}_P(p) > 0$ and choose an integer $\bar{e}_P > e_P/p - 1$. Denoting also by P the maximal ideal of the local ring A at P , then $A/P^{\bar{e}_P}$ is a finite ring and $\theta + P^{\bar{e}_P}$ is a unit of $A/P^{\bar{e}_P}$ so there is a least integer g_P with $\theta^{g_P} \equiv 1 \pmod{P^{\bar{e}_P}}$. Setting $a_P = \text{ord}_P(\theta^{g_P} - 1) \geq \bar{e}_P$, an easy induction (using the binomial expansion of $(1 + \theta^{g_P p^j} - 1)^p$) shows that $\text{ord}_P(\theta^{g_P p^j} - 1) = a_P + e_P j$ for $j \geq 0$. For general k we have $\text{ord}_P(\theta^k - 1) < \bar{e}_P (\leq a_P)$ unless g_P divides k when we can write $k = g_P p^j k_0$ with $p \nmid k_0$: then $(*)$ follows from $e_P j \leq \text{ord}_P k$ and $\text{ord}_P(\theta^k - 1) \leq \text{ord}_P(\theta^{g_P p^j} - 1)$.

Suppose $P \in S_\infty$: for $z \in \mathbb{C}$ we have

$$\lim_{k \rightarrow \infty} |z^k - 1|^{1/k} = \begin{cases} |z|, & \text{if } |z| > 1, \\ 1, & \text{if } |z| < 1. \end{cases}$$

Putting $S_* = \{P \in S_\infty : \|\theta\|_P = 1\}$, we have

$$\lim_{k \rightarrow \infty} \|\lambda_k\|_P^{1/k} = \lim_{k \rightarrow \infty} \frac{\|\theta^{qk} - 1\|_P^{1/k}}{\|\theta^k - 1\|_P^{1/k}} = \max(1, \|\theta\|_P)^{q-1}$$

for $P \notin S_*$, again verifying the claim for all large k .

Finally, each $P \in S_*$ defines a monomorphism $\mathbf{Q}(\theta) \rightarrow \mathbb{C}$ so that $\theta \rightarrow e^{2\pi i \phi_P}$ for some $\phi_P \in \mathbf{R}/\mathbf{Z}$. Now setting $\langle x \rangle = \min_{m \in \mathbf{Z}} |x - m|$ for real x defines a function $\mathbf{R}/\mathbf{Z} \rightarrow [0, \frac{1}{2}]$. Since S_* is finite, Dirichlet's theorem on simultaneous approximation shows that there are infinitely many k so that

$$0 < \langle k\phi_P \rangle < \frac{1}{2q} \quad \text{for all } P \in S_*,$$

and then, clearly, $\langle qk\phi_P \rangle = q\langle k\phi_P \rangle$. Then by the identity

$$|e^{2\pi i x} - 1|^2 = 4 \sin^2 \pi \langle x \rangle, \quad x \in \mathbf{R}/\mathbf{Z},$$

we get

$$\|\lambda_k\|_P = \left(\frac{\sin \pi \langle qk\phi_P \rangle}{\sin \pi \langle k\phi_P \rangle} \right)^2 \geq \left(\frac{2\pi^{-1} \cdot \pi q \langle k\phi_P \rangle}{\pi \langle k\phi_P \rangle} \right)^2 = \left(\frac{2q}{\pi} \right)^2 > 1$$

for all $P \in S_*$ and infinitely many k . This verifies the claim and concludes the proof of the lemma.

REMARK. By using the Tchebotarev Density Theorem it is possible to prove the lemma also for some infinite sets π , namely those of small (upper) Dirichlet density.

PROOF OF THEOREM A. Let π be a nonempty finite set of primes and G a finitely generated solvable group with the π -isolator property.

Case 1. Suppose G is polycyclic. We may assume, if necessary, that G has a finite series with infinite cyclic factors. Using induction on the length of this series we may assume that $G = \langle N, t \rangle$, where N is a nilpotent-by-finite normal subgroup of G . If G is not nilpotent by finite, there is a section of G of the form $J = \langle a, t \rangle$ where $\langle a^J \rangle = A$ is abelian, $J = A \rtimes \langle t \rangle$, and J is not nilpotent-by-finite. By the Key Lemma, J does not have the π -isolator property. This gives the required contradiction.

Case 2. Reduction to Case 1. We will use induction on the solvability length of G . If G is abelian then we are done. Let G be solvable of length $d + 1$ and assume the result holds for groups of length d ($d \geq 1$). Let $A = G^{(d)}$, the d th term of the derived series. Then G/A is polycyclic-by-finite, and by replacing G with a subgroup of finite index, if necessary, we may assume that G/A is polycyclic. Then by the well-known result of P. Hall (see [1]), G satisfies max- n , the maximal condition on normal subgroups. If G is not polycyclic, then there exists $a \in A$ and $t \in G$ such that $\langle a^J \rangle$ is not finitely generated where $J = \langle a, t \rangle$. This group is abelian-by-cyclic. We may assume $G = \langle a, t \rangle$ and $A = \langle a^G \rangle$. If the Prüfer rank of A is not finite,

then G has a section $\langle b \rangle \text{ wr } \langle t \rangle$ isomorphic to the wreath product $C_p \text{ wr } C_\infty$ where p is a prime and C_n denotes the cyclic group of order n . But the group $\langle b \rangle \text{ wr } \langle t \rangle$ does not have the π -isolator property as can be seen by taking $\sqrt[\pi]{H}$ where $H = \langle t \rangle$ if $p \in \pi$ and $H = \langle b, t^q \rangle$ if $p \notin \pi$ where q is any prime in π . Conclude that $G = \langle a, t \rangle$ has finite rank. Now if A is periodic, then it is finite and G is nilpotent-by-finite. We can now invoke the Key Lemma to eliminate the remaining case. This leaves us with the case G is polycyclic and we are in Case 1.

The converse is immediate from P. Hall's result that a nilpotent group has the π -isolator property for all π .

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