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SOLVABLE GROUPS WITH π-ISOLATORS

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ABSTRACT. Let π be any nonempty set of prime numbers. A natural number is a π-number precisely if all of its prime factors are in π. A group G is said to have the π-isolator property if for every subgroup H of G, the set $\sqrt[\pi]{H} = \{g \in G;\ g^n \in H$, for some π-number $n\}$ is a subgroup of G. It is well known that nilpotent groups have the π-isolator property for any nonempty set π of primes. Finitely generated solvable linear groups with finite Prüfer rank, and in particular polycyclic groups, have subgroups of finite index with the π-isolator property if π is the set of all primes. It is shown here that if π is any finite nonempty set of primes and G is a finitely generated solvable group, then G has a subgroup of finite index with the π-isolator property if and only if G is nilpotent-by-finite.

1. Introduction. Let π be a nonempty set of prime numbers. A natural number is a *π-number* precisely if all of its prime factors are in π. A group G is said to have the *π-isolator property* if for every subgroup H of G, the set $\sqrt[\pi]{H} = \{g \in G;\ g^n \in H$ for some π-number $n\}$ is a subgroup of G. A subgroup H of G is called *π-isolated* if $\sqrt[\pi]{H} = H$. The *π-isolator* of a subgroup H of G is the smallest π-isolated subgroup of G containing H. This is the intersection of all π-isolated subgroups of G containing H. The relation $\underset{\pi}{\sim}$ on the set of all subgroups of G given by $K \underset{\pi}{\sim} H$ if and only if $\sqrt[\pi]{K} = \sqrt[\pi]{H}$; $K, H \leq G$, is an equivalence relation. Thus if G has the π-isolator property then each of the equivalence classes has a unique maximal member.

If π is the set of all primes, then the π-isolator of H in G is called the *isolator of H in G* and G is said to possess *the isolator property* if G has the π-isolator property where π is the set of all primes. It was shown in **[2]** that for finitely generated solvable groups those with the isolator property are closely linked to groups with finite (Prüfer) rank. For instance, a torsion-free solvable group of finite rank has a subgroup of finite index with the isolator property. Conversely, if G is a finitely generated nilpotent-by-abelian group with the isolator property, then G has finite rank.

A group G is said to have the *strong isolator property* if it has the isolator property and, in addition, $|\sqrt{H} : H|$ is finite for all subgroups H of G. A polycyclic group has a subgroup of finite index with the strong isolator property. Conversely, if G is a finitely generated solvable group with the strong isolator property, then G is polycyclic (see **[2**, Theorem A]).

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If G is nilpotent (or even locally nilpotent) then it has the π-isolator property for every nonempty set π of primes. On the other hand, if a finitely generated solvable group G has p-isolator property for every prime p, then G is necessarily nilpotent. This follows from three observations. (i) A finite group of this type is nilpotent. (ii) If a finitely generated solvable group is not nilpotent then it has a finite quotient which is not nilpotent (see [**3**, Theorem 10.51]). (iii) The π-isolator property is quotient closed. In this paper we investigate solvable groups G which have the π-isolator property for some finite (nonempty) set π of primes. It would be conceivable that polycyclic groups would perhaps have this property. The answer is to the contrary and our main result is the following.

THEOREM A. *Let π be a nonempty finite set of primes, and let G be a finitely generated solvable group. Then G has a subgroup of finite index possessing the π-isolator property if and only if G is nilpotent-by-finite.*

2. Proofs. The proof of Theorem A is not direct. We find it desirable to start with the following key lemma. If $\theta \neq 0$ is algebraic over $\mathbf{Q}$, let $\mathbf{Z}\langle\theta\rangle$ denote the subring of $\mathbf{Q}(\theta)$ generated by $1, \theta, \theta^{-1}$. Define the group Γ_θ as the semidirect product of additive groups $\mathbf{Z}\langle\theta\rangle$, $\mathbf{Z}$. Thus $\Gamma_\theta = \mathbf{Z}\langle\theta\rangle \rtimes \mathbf{Z}$, and

$$(\alpha, n)\cdot(\beta, m) = (\alpha + \theta^n\beta, n+m); \qquad \alpha, \beta \in \mathbf{Z}\langle\theta\rangle, n, m \in \mathbf{Z}.$$

KEY LEMMA. *If π is a finite set of prime numbers so that Γ_θ has the π-isolator property, then either*

(a) *π is empty, or*

(b) *θ is a root of unity.*

PROOF. Suppose, by way of contradiction, that Γ_θ has the π-isolator property, θ is not a root of unity, and π has a largest element q. For $k \geqslant 1$ put

$$\lambda_k = 1 + \theta^k + \theta^{2k} + \cdots + \theta^{(q-1)k} = \frac{\theta^{kq}-1}{\theta^k - 1} \neq 0 \quad \text{in } \mathbf{Z}\langle\theta\rangle.$$

The prime divisors P of $\mathbf{Q}(\theta)$ have corresponding absolute values

$$\|\alpha\|_P = \begin{cases} \operatorname{norm}(P)^{-\operatorname{ord}_P(\alpha)}, & P \text{ non-Archimedean},\\ |\alpha|, & P \text{ real},\\ |\alpha|^2, & P \text{ complex},\end{cases}$$

normalized so that the Product Formula holds:

$$\prod_P \|\alpha\|_P = 1 \quad \text{for } 0 \neq \alpha \in \mathbf{Q}(\theta).$$

Since θ is not a root of unity it has height

$$H = \prod_P \max(1, \|\theta\|_P) > 1$$

(see, for example, Theorem 8 on p. 77 of [4]). Consider the following classes of prime divisors of $\mathbf{Q}(\theta)$:

S_∞: Archimedean primes,
S_θ: $P \notin S_\infty$ so $\operatorname{ord}_P \theta \neq 0$,
S_π: $P \notin S_\infty \cup S_\theta$ so P lies above some $p \in \pi$.

Then $S = S_\infty \cup S_\theta \cup S_\pi$ is finite and for $\varepsilon > 0$ we *claim*

$$\frac{\|\lambda_k\|_P^{1/k}}{\max(1,\|\theta\|_P)^{q-1}} \begin{cases} = 1 & \text{for } P \notin S \text{ and all } k, \\ \geqslant 1-\varepsilon & \text{for } P \in S \text{ and infinitely many } k. \end{cases}$$

The lemma is an immediate consequence since the claim shows that

$$\frac{1}{H^{q-1}} = \prod_P \frac{\|\lambda_k\|_P^{1/k}}{\max(1,\|\theta\|_P)^{q-1}} \geqslant (1-\varepsilon)^{\operatorname{card} S},$$

for infinitely many k, holds for each $\varepsilon > 0$. Clearly, we may choose $\varepsilon > 0$ to contradict $H > 1$ so it remains only to verify the claim.

Suppose $P \notin S$: since $\|\theta\|_P = 1$ and $\lambda_k \in \mathbf{Z}\langle\theta\rangle$, we need only rule out the possibility that $\operatorname{ord}_P \lambda_k > 0$. If this was the case then λ_k is in the ideal $P_0 = \{\alpha \in \mathbf{Z}\langle\theta\rangle : \operatorname{ord}_P \alpha > 0\}$ of $\mathbf{Z}\langle\theta\rangle$ so (λ_k, kq) is in the subgroup $H = P_0 \rtimes kq\mathbf{Z}$ of Γ_θ. By $(0,-k)^q = (0,-kq)$, $(1,k)^q = (\lambda_k, kq)$ both $(0,-k)$, $(1,k)$ are in $\sqrt[\pi]{H}$ and since this is a group, by hypothesis, it follows that $(1,0) = (1,k)(0,-k)$ is in $\sqrt[\pi]{H}$. This means that $(N,0) = (1,0)^N$ is in $H \cap \mathbf{Z}\langle\theta\rangle = P_0$ for some π-number N, hence that $\operatorname{ord}_P N > 0$ contrary to $P \notin S$.

Suppose $P \in S_\theta$: here the claim follows for all $k \geqslant 1$ from

$$\lambda_k = \frac{1-\theta^{kq}}{1-\theta^k} \quad \text{if } \|\theta\|_P < 1 \quad \text{and} \quad \lambda_k = \theta^{k(q-1)}\frac{1-\theta^{-kq}}{1-\theta^{-k}} \quad \text{if } \|\theta\|_P > 1.$$

Suppose $P \in S_\pi$: it suffices to show that there is a constant a_P depending only on P (and θ) so that

$$(*) \qquad \operatorname{ord}_P(\theta^k - 1) \leqslant a_P + \operatorname{ord}_P(k) \quad \text{for all } k \geqslant 1.$$

For then

$$\|\lambda_k\|_P^{1/k} = \left\| \frac{\theta^{kq}-1}{\theta^k - 1} \right\|_P^{1/k} \geqslant \|\theta^{kq}-1\|_P^{1/k} \geqslant \operatorname{norm}(P)^{-(a_P + \operatorname{ord}_P(kq))/k}$$

which is $> 1-\varepsilon$ for *all* large enough k.

So let p be the prime number below P, set $e_P = \operatorname{ord}_P(p) > 0$ and choose an integer $\bar{e}_P > e_P/p - 1$. Denoting also by P the maximal ideal of the local ring A at P, then $A/P^{\bar{e}_P}$ is a finite ring and $\theta + P^{\bar{e}_P}$ is a unit of $A/P^{\bar{e}_P}$ so there is a least integer g_P with $\theta^{g_P} \equiv 1 \bmod P^{\bar{e}_P}$. Setting $a_P = \operatorname{ord}_P(\theta^{g_P} - 1) \geqslant \bar{e}_P$, an easy induction (using the binomial expansion of $(1 + \theta^{g_P p^j} - 1)^p$) shows that $\operatorname{ord}_P(\theta^{g_P p^j} - 1) = a_P + e_P j$ for $j \geqslant 0$. For general k we have $\operatorname{ord}_P(\theta^k - 1) < \bar{e}_P (\leqslant a_P)$ unless g_P divides k when we can write $k = g_P p^j k_0$ with $p \nmid k_0$: then $(*)$ follows from $e_P j \leqslant \operatorname{ord}_P k$ and $\operatorname{ord}_P(\theta^k - 1) \leqslant \operatorname{ord}_P(\theta^{g_P p^j} - 1)$.

Suppose $P \in S_\infty$: for $z \in \mathbf{C}$ we have

$$\lim_{k\to\infty} |z^k - 1|^{1/k} = \begin{cases} |z|, & \text{if } |z| > 1, \\ 1, & \text{if } |z| < 1. \end{cases}$$

Putting $S_* = \{P \in S_\infty : \|\theta\|_P = 1\}$, we have

$$\lim_{k\to\infty} \|\lambda_k\|_P^{1/k} = \lim_{k\to\infty} \frac{\|\theta^{qk} - 1\|_P^{1/k}}{\|\theta^k - 1\|_P^{1/k}} = \max(1, \|\theta\|_P)^{q-1}$$

for $P \notin S_*$, again verifying the claim for *all* large k.

Finally, each $P \in S_*$ defines a monomorphism $\mathbf{Q}(\theta) \to \mathbf{C}$ so that $\theta \to e^{2\pi i \phi_P}$ for some $\phi_P \in \mathbf{R}/\mathbf{Z}$. Now setting $\langle x \rangle = \min_{m \in Z} |x - m|$ for real x defines a function $\mathbf{R}/\mathbf{Z} \to [0, \frac{1}{2}]$. Since S_* is finite, Dirichlet's theorem on simultaneous approximation shows that there are infinitely many k so that

$$0 < \langle k\phi_P \rangle < \frac{1}{2q} \quad \text{for all } P \in S_*,$$

and then, clearly, $\langle qk\phi_P \rangle = q\langle k\phi_P \rangle$. Then by the identity

$$|e^{2\pi i x} - 1|^2 = 4\sin^2 \pi \langle x \rangle, \qquad x \in \mathbf{R}/\mathbf{Z},$$

we get

$$\|\lambda_k\|_P = \left(\frac{\sin \pi \langle qk\phi_P \rangle}{\sin \pi \langle k\phi_P \rangle} \right)^2 \geqslant \left(\frac{2\pi^{-1} \cdot \pi q \langle k\phi_P \rangle}{\pi \langle k\phi_P \rangle} \right)^2 = \left(\frac{2q}{\pi} \right)^2 > 1$$

for all $P \in S_*$ and infinitely many k. This verifies the claim and concludes the proof of the lemma.

REMARK. By using the Tchebotarev Density Theorem it is possible to prove the lemma also for some infinite sets π, namely those of small (upper) Dirichlet density.

PROOF OF THEOREM A. Let π be a nonempty finite set of primes and G a finitely generated solvable group with the π-isolator property.

Case 1. Suppose G is polycyclic. We may assume, if necessary, that G has a finite series with infinite cyclic factors. Using induction on the length of this series we may assume that $G = \langle N, t \rangle$, where N is a nilpotent-by-finite normal subgroup of G. If G is not nilpotent by finite, there is a section of G of the form $J = \langle a, t \rangle$ where $\langle a^J \rangle = A$ is abelian, $J = A \rtimes \langle t \rangle$, and J is not nilpotent-by-finite. By the Key Lemma, J does not have the π-isolator property. This gives the required contradiction.

Case 2. Reduction to Case 1. We will use induction on the solvability length of G. If G is abelian then we are done. Let G be solvable of length $d + 1$ and assume the result holds for groups of length d ($d \geqslant 1$). Let $A = G^{(d)}$, the dth term of the derived series. Then G/A is polycyclic-by-finite, and by replacing G with a subgroup of finite index, if necessary, we may assume that G/A is polycyclic. Then by the well-known result of P. Hall (see [**1**]), G satisfies max-n, the maximal condition on normal subgroups. If G is not polycyclic, then there exists $a \in A$ and $t \in G$ such that $\langle a^J \rangle$ is not finitely generated where $J = \langle a, t \rangle$. This group is abelian-by-cyclic. We may assume $G = \langle a, t \rangle$ and $A = \langle a^G \rangle$. If the Prüfer rank of A is not finite, then G has a section $\langle b \rangle \operatorname{wr} \langle t \rangle$ isomorphic to the wreath product $C_p \operatorname{wr} C_\infty$ where p is a prime and C_n denotes the cyclic group of order n. But the group $\langle b \rangle \operatorname{wr} \langle t \rangle$ does not have the π-isolator property as can be seen by taking $\sqrt[\pi]{H}$ where $H = \langle t \rangle$ if $p \in \pi$ and $H = \langle b, t^q \rangle$ if $p \notin \pi$ where q is any prime in π. Conclude that $G = \langle a, t \rangle$ has finite rank. Now if A is periodic, then it is finite and G is nilpotent-by-finite. We can now invoke the Key Lemma to eliminate the remaining case. This leaves us with the case G is polycyclic and we are in Case 1.

The converse is immediate from P. Hall's result that a nilpotent group has the π-isolator property for all π.

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