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AUTOMORPHISM GROUPS OF RULED FUNCTION FIELDS AND A PROBLEM OF ZARISKI

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ABSTRACT. Let K_1 and K_2 be finitely generated extensions of a field K and let x be transcendental over K_1 and K_2, and assume $K_1(x) = K_2(x)$. The main results show that if K is infinite and the group of automorphisms of K_2 over K is finite, or if K is finite and the group of automorphisms of $\bar{K}K_2$ over $\bar{K}$ ($\bar{K}$ the algebraic closure of K) is finite, then K_1 equals K_2.

Let K_1 and K_2 be finitely generated extensions of a field K and let x_i be transcendental over K_i, $i = 1, 2$. The Zariski problem [**4**] asks if $K_1(x_1) = K_2(x_2)$ must K_1 and K_2 be K-isomorphic. Some special cases of this problem have been solved [**1**, **4**], but in general the problem is open. In this paper we improve some known results and establish an affirmative answer for a new class of fields in a special case.

DEFINITION 1. Let L be a finitely generated extension of a field K. If K is infinite and the group of K-automorphisms of L is finite, $|\operatorname{aut}_K L| < \infty$, then L is of general type over K. If K is finite and $|\operatorname{aut}_{\bar{K}} L\bar{K}| < \infty$ for $\bar{K}$ an algebraic closure of K, then L is of general type over K.

The motivation for this definition is the paper of Husemoller [**3**]. He discusses the canonical dimension of a variety over an algebraically closed field. He defines an r-dimensional variety V to be of general type provided the canonical dimension of V is r (which for example, includes curves of genus greater than 1). He then goes on to show that if $k(v)$ is the function field of a variety of general type, then the group of k-automorphisms of $k(v)$ is finite. The special definition for K finite is to prevent $K(x)$ from being of general type over K. The main results of this paper related to the Zariski problem assert that if K_2 is of general type over K, and $x_1 = x_2$, then K_1 equals K_2. For the case of an infinite base field K, Samuel [**4**] has shown K_1 and K_2 are K-isomorphic. For the case of a finite base field, nothing had been known.

We first make a few general observations. In order to achieve an affirmative answer to the Zariski problem, one can assume $K_1 \cap K_2 = K$. Thus one can assume K is algebraically closed in $K_1(x_1)$. Furthermore, since $K_1(x_1)$ is separable over K_1 and K_2, it is separable over their intersection [**2**, Theorem 1.1, p. 1304], and hence each K_i is separable over K, i.e. is regular over K. We note that if $\bar{K}$ denotes the algebraic closure of K, and K_2 is regular over K, then every K-automorphism of K_2

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has a unique extension to a $\bar{K}$-automorphisms of $K_2\bar{K} = K_2 \otimes_K \bar{K}$. Thus if $K_2\bar{K}$ is the function field of a variety of general type over $\bar{K}$, and K_2 is regular over K, K_2 is of general type over K. Part of the proof of Theorem 2 is essentially due to Roquette [**5**, Lemma 1, p. 209].

THEOREM 2. *Assume $L = K_1(x) = K_2(x) \supset K$ where x is transcendental over K_i, and K_i is a finitely generated extension of an infinite field K, $i = 1, 2$. If the group of K-automorphisms of K_2 is finite, then $K_1 = K_2$.*

PROOF. Let $\{w_1, w_2, \ldots, w_r\}$ generate K_2 over K. Each of these is a rational function $f_i(x)/g_i(x)$ in x with coefficients in K_1. Let the nonzero coefficients of $f_i(x)$ be $\{a_{ij}\}$ and the nonzero coefficients of $g_i(x)$ be $\{b_{ij}\}$. Let $a_i = f_i(x)$, $b_i = g_i(x)$. Each of the elements of $\{a_i, a_{ij}, b_i, b_{ij}\}$ is also a rational function in x with coefficients in K_2. There are only a finite number of prime divisors of $K_2(x)$ over K_2 for which the associated place is either 0 or ∞ at any given element. Since $|K| = \infty$, there is an infinite number of elements $\alpha_i \in K$ such that the $(x - \alpha_i)$-place of $K_2(x)$ onto K_2 is finite and nonzero at each $\{a_i, a_{ij}, b_i, b_{ij}\}$. Thus, for each of these places $p_{(x-\alpha_i)}$,

$$p_{(x-\alpha_i)(w_j)} = p_{(x-\alpha_i)}\left(\frac{f_j(x)}{g_j(x)}\right) = \frac{p_{(x-\alpha_i)}(f_j)(\alpha_i)}{p_{(x-\alpha_i)}(g_j)(\alpha_i)} \in p_{x-\alpha_i}(K_1).$$

Thus $K_2 \subseteq p_{x-\alpha_i}(K_1)$, i.e. $K_2 = p_{x-\alpha_i}(K_1)$. Thus we have an infinite number of elements $\{\alpha_i\}$ of K such that the $p_{x-\alpha_i}$ place of $K_2(x)$ over K_2 gives a K-isomorphism of K_1 onto K_2. Symmetrically, we can certainly get a single K-isomorphism σ: $K_2 \to K_1$.

We now assume there exists an element, z, of K_1 which is not an element of K_2 and we get a contradiction. $z = r(x)$ is a nonconstant rational function in x with coefficients in K_2. Choose $\alpha_0 \in \{\alpha_i\}$ as above and consider $r(x) - r(\alpha_0)$, which is also a nonconstant rational function with coefficients in K_2. As noted above, each $p_{(x-\alpha_i)}$ defines a K-isomorphism of K_1 onto K_2. Call this isomorphism $\bar{\alpha}_i$. Then each $\bar{\alpha}_i \circ \sigma$ defines a K-automorphism of K_2. Since the group of K-automorphisms of K_2 is finite, and $\bar{\alpha}_i \circ \sigma = \bar{\alpha}_j \circ \sigma$ if and only if the isomorphisms $\bar{\alpha}_i = \bar{\alpha}_j$, there must be some infinite family of automorphisms $\bar{\alpha}_i$ which are equal. We may assume α_0 is in this family. But then each of the elements associated to the automorphisms must be a root of the nonzero rational function $r(x) - r(\alpha_0)$. But this is a contradiction since a nonzero rational function has only a finite number of roots. Thus $K_1 = K_2$.

Let K be a field and let $\{x, y\}$ be algebraically independent over K. Note that $K(y)(x) = K(y - x)(x)$, and yet $K(y) \neq K(y - x)$. Moreover, if K is finite, $|\mathrm{aut}_K K(y - x)| < \infty$. Thus neither of the assumptions in the theorem is superfluous.

COROLLARY 3. *Assume K is infinite, the group of K-automorphisms of K_1 is finite and x is transcendental over K_1. Then the natural injection σ: $\mathrm{aut}_K K_1 \to \mathrm{aut}_{K(x)} K_1(x)$ is also surjective.*

PROOF. Clearly every K-automorphism θ of K_1 can be uniquely extended to a $K(x)$-automorphism of $K_1(x)$ by defining $\theta(x) = x$. Now let θ be any $K(x)$-automorphism of $K_1(x)$. Then $K_1(x) = K_1^\theta(\theta(x)) = K_1^\theta(x)$, where K_1^θ denotes the image of K_1 under θ. By Theorem 2, $K_1 = K_1^\theta$, i.e. θ is an extension of a K-automorphism of K_1.

For $K(y)$ a simple transcendental extension of K, the $K(x)$-automorphism of $K(y, x)$ which sends y to $y + x$ will not be the extension of any K-automorphism of $K(y)$.

LEMMA 4. *Let K_1 and K_2 be subfields of a field L and assume $K_1 \cap K_2 = K$. If L and F are linearly disjoint over K, and $K_1F = K_2F$, then $K_1 = K_2$.*

PROOF. By the standard lemma on linear disjointness, K_iF and L are linearly disjoint over K_i, and hence $K_iF \cap L = K_i$, $i = 1, 2$. Thus if $K_1F = K_2F$, $K_1 = K_2$.

THEOREM 5. *Assume we have $L = K_1(x) = K_2(x) \supset K$ where x is transcendental over K_i, and K_i is a finitely generated extension of a finite field K. If K_2 is of general type over K, then $K_1 = K_2$.*

PROOF. If $K_1 \cap K_2$ is not a finite field, then we may apply Theorem 2. Thus we may assume $K_1 \cap K_2 = K$, and hence K is algebraically closed in $K_1(x)$. Since K is perfect, $K_1(x)$ is regular over K. Thus $K_1(x)$ is linearly disjoint over K from $\bar{K}$, the algebraic closure of K. Since K_2/K is of general type, Theorem 2 asserts $K_1\bar{K} = K_2\bar{K}$. By Lemma 4, $K_1 = K_2$.

It should be noted that Theorem 5 is true under slightly more general conditions. For example, if $|\operatorname{aut}_{K(y)} K_2(y)| < \infty$, where y is transcendental over $K_2(x)$, then a similar application of Theorem 2 and Lemma 4 shows $K_1 = K_2$.

The main results of this paper are related to [**4**, Theorem 2, p. 87 and Corollary 2, p. 88]. Nagata uses the hypothesis (N) that no algebraic extension of K_2 is ruled over K, whereas the present paper uses the hypothesis (D) that $|\operatorname{Aut}_K(K_2)| < \infty$. Consider the 1-dimensional case, i.e., $\operatorname{tr.d.}(K_2/K) = 1$. For K of characteristic 0, the fields satisfying (D) are exactly those of genus $\geqslant 2$, while those satisfying (N) are exactly those of genus $\geqslant 1$. The latter point follows since a separable base charge cannot lower the genus. Thus in this case Nagata's theorem implies the present result. If the characteristic of K is $p \neq 0$, then there exist examples of curves of genus $\geqslant 2$ such that base charge drops the genus to 0, e.g. let $y^2 = x^p - a$, $a^{1/p} \notin K$, and adjoint $a^{1/p}$ to $K(x, y)$. For these curves, (N) does not hold but (D) does. The author is indebted to the referee for the above comments.

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