

## AUTOMORPHISM GROUPS OF RULED FUNCTION FIELDS AND A PROBLEM OF ZARISKI

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**ABSTRACT.** Let  $K_1$  and  $K_2$  be finitely generated extensions of a field  $K$  and let  $x$  be transcendental over  $K_1$  and  $K_2$ , and assume  $K_1(x) = K_2(x)$ . The main results show that if  $K$  is infinite and the group of automorphisms of  $K_2$  over  $K$  is finite, or if  $K$  is finite and the group of automorphisms of  $\bar{K}K_2$  over  $\bar{K}$  ( $\bar{K}$  the algebraic closure of  $K$ ) is finite, then  $K_1$  equals  $K_2$ .

Let  $K_1$  and  $K_2$  be finitely generated extensions of a field  $K$  and let  $x_i$  be transcendental over  $K_i$ ,  $i = 1, 2$ . The Zariski problem [4] asks if  $K_1(x_1) = K_2(x_2)$  must  $K_1$  and  $K_2$  be  $K$ -isomorphic. Some special cases of this problem have been solved [1, 4], but in general the problem is open. In this paper we improve some known results and establish an affirmative answer for a new class of fields in a special case.

**DEFINITION 1.** Let  $L$  be a finitely generated extension of a field  $K$ . If  $K$  is infinite and the group of  $K$ -automorphisms of  $L$  is finite,  $|\text{aut}_K L| < \infty$ , then  $L$  is of general type over  $K$ . If  $K$  is finite and  $|\text{aut}_{\bar{K}} L\bar{K}| < \infty$  for  $\bar{K}$  an algebraic closure of  $K$ , then  $L$  is of general type over  $K$ .

The motivation for this definition is the paper of Husemoller [3]. He discusses the canonical dimension of a variety over an algebraically closed field. He defines an  $r$ -dimensional variety  $V$  to be of general type provided the canonical dimension of  $V$  is  $r$  (which for example, includes curves of genus greater than 1). He then goes on to show that if  $k(v)$  is the function field of a variety of general type, then the group of  $k$ -automorphisms of  $k(v)$  is finite. The special definition for  $K$  finite is to prevent  $K(x)$  from being of general type over  $K$ . The main results of this paper related to the Zariski problem assert that if  $K_2$  is of general type over  $K$ , and  $x_1 = x_2$ , then  $K_1$  equals  $K_2$ . For the case of an infinite base field  $K$ , Samuel [4] has shown  $K_1$  and  $K_2$  are  $K$ -isomorphic. For the case of a finite base field, nothing had been known.

We first make a few general observations. In order to achieve an affirmative answer to the Zariski problem, one can assume  $K_1 \cap K_2 = K$ . Thus one can assume  $K$  is algebraically closed in  $K_1(x_1)$ . Furthermore, since  $K_1(x_1)$  is separable over  $K_1$  and  $K_2$ , it is separable over their intersection [2, Theorem 1.1, p. 1304], and hence each  $K_i$  is separable over  $K$ , i.e. is regular over  $K$ . We note that if  $\bar{K}$  denotes the algebraic closure of  $K$ , and  $K_2$  is regular over  $K$ , then every  $K$ -automorphism of  $K_2$

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has a unique extension to a  $\bar{K}$ -automorphisms of  $K_2\bar{K} = K_2 \otimes_K \bar{K}$ . Thus if  $K_2\bar{K}$  is the function field of a variety of general type over  $\bar{K}$ , and  $K_2$  is regular over  $K$ ,  $K_2$  is of general type over  $K$ . Part of the proof of Theorem 2 is essentially due to Roquette [5, Lemma 1, p. 209].

**THEOREM 2.** *Assume  $L = K_1(x) = K_2(x) \supset K$  where  $x$  is transcendental over  $K_i$ , and  $K_i$  is a finitely generated extension of an infinite field  $K$ ,  $i = 1, 2$ . If the group of  $K$ -automorphisms of  $K_2$  is finite, then  $K_1 = K_2$ .*

**PROOF.** Let  $\{w_1, w_2, \dots, w_r\}$  generate  $K_2$  over  $K$ . Each of these is a rational function  $f_i(x)/g_i(x)$  in  $x$  with coefficients in  $K_1$ . Let the nonzero coefficients of  $f_i(x)$  be  $\{a_{ij}\}$  and the nonzero coefficients of  $g_i(x)$  be  $\{b_{ij}\}$ . Let  $a_i = f_i(x)$ ,  $b_i = g_i(x)$ . Each of the elements of  $\{a_i, a_{ij}, b_i, b_{ij}\}$  is also a rational function in  $x$  with coefficients in  $K_2$ . There are only a finite number of prime divisors of  $K_2(x)$  over  $K_2$  for which the associated place is either 0 or  $\infty$  at any given element. Since  $|K| = \infty$ , there is an infinite number of elements  $\alpha_i \in K$  such that the  $(x - \alpha_i)$ -place of  $K_2(x)$  onto  $K_2$  is finite and nonzero at each  $\{a_i, a_{ij}, b_i, b_{ij}\}$ . Thus, for each of these places  $P_{(x-\alpha_i)}$ ,

$$P_{(x-\alpha_i)(w_j)} = P_{(x-\alpha_i)} \left( \frac{f_j(x)}{g_j(x)} \right) = \frac{P_{(x-\alpha_i)}(f_j)(\alpha_i)}{P_{(x-\alpha_i)}(g_j)(\alpha_i)} \in P_{x-\alpha_i}(K_1).$$

Thus  $K_2 \subseteq P_{x-\alpha_i}(K_1)$ , i.e.  $K_2 = P_{x-\alpha_i}(K_1)$ . Thus we have an infinite number of elements  $\{\alpha_i\}$  of  $K$  such that the  $P_{x-\alpha_i}$  place of  $K_2(x)$  over  $K_2$  gives a  $K$ -isomorphism of  $K_1$  onto  $K_2$ . Symmetrically, we can certainly get a single  $K$ -isomorphism  $\sigma: K_2 \rightarrow K_1$ .

We now assume there exists an element,  $z$ , of  $K_1$  which is not an element of  $K_2$  and we get a contradiction.  $z = r(x)$  is a nonconstant rational function in  $x$  with coefficients in  $K_2$ . Choose  $\alpha_0 \in \{\alpha_i\}$  as above and consider  $r(x) - r(\alpha_0)$ , which is also a nonconstant rational function with coefficients in  $K_2$ . As noted above, each  $P_{(x-\alpha_i)}$  defines a  $K$ -isomorphism of  $K_1$  onto  $K_2$ . Call this isomorphism  $\bar{\alpha}_i$ . Then each  $\bar{\alpha}_i \circ \sigma$  defines a  $K$ -automorphism of  $K_2$ . Since the group of  $K$ -automorphisms of  $K_2$  is finite, and  $\bar{\alpha}_i \circ \sigma = \bar{\alpha}_j \circ \sigma$  if and only if the isomorphisms  $\bar{\alpha}_i = \bar{\alpha}_j$ , there must be some infinite family of automorphisms  $\bar{\alpha}_i$  which are equal. We may assume  $\alpha_0$  is in this family. But then each of the elements associated to the automorphisms must be a root of the nonzero rational function  $r(x) - r(\alpha_0)$ . But this is a contradiction since a nonzero rational function has only a finite number of roots. Thus  $K_1 = K_2$ .

Let  $K$  be a field and let  $\{x, y\}$  be algebraically independent over  $K$ . Note that  $K(y)(x) = K(y - x)(x)$ , and yet  $K(y) \neq K(y - x)$ . Moreover, if  $K$  is finite,  $|\text{aut}_K K(y - x)| < \infty$ . Thus neither of the assumptions in the theorem is superfluous.

**COROLLARY 3.** *Assume  $K$  is infinite, the group of  $K$ -automorphisms of  $K_1$  is finite and  $x$  is transcendental over  $K_1$ . Then the natural injection  $\sigma: \text{aut}_K K_1 \rightarrow \text{aut}_{K(x)} K_1(x)$  is also surjective.*

PROOF. Clearly every  $K$ -automorphism  $\theta$  of  $K_1$  can be uniquely extended to a  $K(x)$ -automorphism of  $K_1(x)$  by defining  $\theta(x) = x$ . Now let  $\theta$  be any  $K(x)$ -automorphism of  $K_1(x)$ . Then  $K_1(x) = K_1^\theta(\theta(x)) = K_1^\theta(x)$ , where  $K_1^\theta$  denotes the image of  $K_1$  under  $\theta$ . By Theorem 2,  $K_1 = K_1^\theta$ , i.e.  $\theta$  is an extension of a  $K$ -automorphism of  $K_1$ .

For  $K(y)$  a simple transcendental extension of  $K$ , the  $K(x)$ -automorphism of  $K(y, x)$  which sends  $y$  to  $y + x$  will not be the extension of any  $K$ -automorphism of  $K(y)$ .

LEMMA 4. Let  $K_1$  and  $K_2$  be subfields of a field  $L$  and assume  $K_1 \cap K_2 = K$ . If  $L$  and  $F$  are linearly disjoint over  $K$ , and  $K_1F = K_2F$ , then  $K_1 = K_2$ .

PROOF. By the standard lemma on linear disjointness,  $K_iF$  and  $L$  are linearly disjoint over  $K_i$ , and hence  $K_iF \cap L = K_i$ ,  $i = 1, 2$ . Thus if  $K_1F = K_2F$ ,  $K_1 = K_2$ .

THEOREM 5. Assume we have  $L = K_1(x) = K_2(x) \supset K$  where  $x$  is transcendental over  $K_i$ , and  $K_i$  is a finitely generated extension of a finite field  $K$ . If  $K_2$  is of general type over  $K$ , then  $K_1 = K_2$ .

PROOF. If  $K_1 \cap K_2$  is not a finite field, then we may apply Theorem 2. Thus we may assume  $K_1 \cap K_2 = K$ , and hence  $K$  is algebraically closed in  $K_1(x)$ . Since  $K$  is perfect,  $K_1(x)$  is regular over  $K$ . Thus  $K_1(x)$  is linearly disjoint over  $K$  from  $\bar{K}$ , the algebraic closure of  $K$ . Since  $K_2/K$  is of general type, Theorem 2 asserts  $K_1\bar{K} = K_2\bar{K}$ . By Lemma 4,  $K_1 = K_2$ .

It should be noted that Theorem 5 is true under slightly more general conditions. For example, if  $|\text{aut}_{K(y)} K_2(y)| < \infty$ , where  $y$  is transcendental over  $K_2(x)$ , then a similar application of Theorem 2 and Lemma 4 shows  $K_1 = K_2$ .

The main results of this paper are related to [4, Theorem 2, p. 87 and Corollary 2, p. 88]. Nagata uses the hypothesis (N) that no algebraic extension of  $K_2$  is ruled over  $K$ , whereas the present paper uses the hypothesis (D) that  $|\text{Aut}_K(K_2)| < \infty$ . Consider the 1-dimensional case, i.e.,  $\text{tr. d.}(K_2/K) = 1$ . For  $K$  of characteristic 0, the fields satisfying (D) are exactly those of genus  $\geq 2$ , while those satisfying (N) are exactly those of genus  $\geq 1$ . The latter point follows since a separable base change cannot lower the genus. Thus in this case Nagata's theorem implies the present result. If the characteristic of  $K$  is  $p \neq 0$ , then there exist examples of curves of genus  $\geq 2$  such that base change drops the genus to 0, e.g. let  $y^2 = x^p - a$ ,  $a^{1/p} \notin K$ , and adjoint  $a^{1/p}$  to  $K(x, y)$ . For these curves, (N) does not hold but (D) does. The author is indebted to the referee for the above comments.

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