COMMUTATIVE FPF RINGS ARISING AS
SPLIT-NUL EXTENSIONS

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Abstract. Let $R = (B, E)$ be the split-null or trivial extension of a faithful module $E$ over a commutative ring $B$. $R$ is an FPF ring if the partial quotient ring $BS^{-1}$ with respect to the set $S$ of elements of $B$ with zero annihilator in $E$ is canonically the endomorphism ring of $E$, that is $BS^{-1} = \text{End}_BE^{-1}$, every finitely generated ideal with zero annihilator in $E$ is invertible in $BS^{-1}$, and $E = ES^{-1}$ is an injective module over $B$. The proof uses the author's characterization of commutative FPF rings [1] and also the characterization of self-injectivity of a split-null extension [3].

Background. A ring $R$ is (right) [F]PF if every [finitely generated] faithful right $R$-module generates the category mod-$R$ of all right $R$-modules. A module $M$ is (Beachy-Blair) cofaithful if there is an embedding $R \rightarrow M^n$ for some finite integer $n$. Over a commutative ring $R$, every finitely generated faithful module is manifestly cofaithful (= CF-faithful in [4]). Furthermore, any cofaithful module over a right self-injective ring $R$ generates mod-$R$, since then $R$ splits in any over module, hence $M^n \cong R \oplus X$ in mod-$R$ for some $n > 0$. This shows that any commutative self-injective ring $R$ is FPF. Moreover, a commutative FPF ring $R$ is characterized in [1] by the two conditions:

(FPF 1) $R$ is quotient-injective, i.e. the (classical) quotient ring $Q_e(R)$ is injective.
(FPF 2) $R$ is pre-FPF, i.e. finitely generated faithful ideals are generators of mod-$R$ (equivalently, are finitely generated projective [1, 2]).

Since any commutative self-injective ring is FPF, it is thereby pre-FPF, that is, (FPF$_1$)$\Rightarrow$(FPF$_2$) for injective $R$: in fact, then $R$ is the only finitely generated faithful ideal.

In [3] we considered the split-null (or trivial) extension $R = (B, E)$ of a faithful $B$-bimodule $E$ over a ring $B$, and characterized the conditions under which $(B, E)$ is right PF (resp. right self-injective). We say that $E$ is left strongly balanced [3] if $B$ is canonically isomorphic to the endomorphism ring of the right $B$-module $E$; notation: $B = \text{End}_BE$. Theorem 2 of [3] states

(IN 1) $(B, E)$ is right self-injective iff $E$ is a left strongly balanced injective right $B$-module.
(IN 2) $(B, E)$ is right PF iff $E$ is a left strongly balanced injective right cogenerator over $B$.

Henceforth let $B$ be a commutative ring, and let $E$ be a faithful $B$-module. Then $R = (B, E)$ is commutative, and our main theorem characterizes when $R$ is FPF. To
describe the result, let $S$ denote the multiplicative set of $B$ consisting of all $b \in B$ such that $\ker b = 0$, i.e., $b : E \to E$ is monic. Then, $S$ is a subset of the set $B^*$ of regular elements of $B$, hence the quotient ring $Q = BS^{-1}$ embeds in $Q = Q_c(B)$ canonically.

1. **Proposition and Definition.** An ideal $I$ of $B$ is said to invert or is invertible in a commutative overring $Q$ provided the equivalent conditions hold:

   (a) $I/I' = B$ for $I' = (I:B) = \{ q \in Q | qI \subseteq B \}$.
   
   (b) $\sum_{i=1}^n q_i b_i = B$ for finitely many elements $q_1, \ldots, q_n \in Q$.
   
   (c) There exist elements $q_i, \ldots, q_n \in Q$, and $b_i, \ldots, b_n \in I$ so that $q_i b_i \in B$, $i = 1, \ldots, n$, and $\sum_{i=1}^n q_i b_i = 1$.
   
   (d) $I$ is a faithful ideal of $B$, and there exist $q_1, \ldots, q_n \in Q$, $b_1, \ldots, b_n \in I$ so that $q_i b_i \in B$, $i = 1, \ldots, n$, and $x = \sum_{i=1}^n b_i q_i(x) \forall x \in I$.
   
   (e) $I$ is a finitely generated faithful projective $B$-module, and if $f \in \text{Hom}_B(I,B)$, then $f = q_i$ for some $q \in Q$, where $q_i(x) = qx \forall x \in B$.
   
   (f) $I$ generates $\text{mod}-B$ and if $f \in \text{Hom}_B(I,B)$, then $f = q_i$ for some $q \in Q$.

   **Proof.** (a) $\iff$ (b) $\iff$ (c) $\iff$ (d) is direct, and (d) $\iff$ (e) uses the dual basis lemma. It can be shown that $b_1, \ldots, b_n$ generates $I$, that $(q_1)_s, \ldots, (q_n)_s$ generates $\text{Hom}_B(I,B)$, and if $f \in \text{Hom}_B(I,B)$, then there exists $c_i \in B$, $i = 1, \ldots, n$, so that
   
   $$f = \sum_{i=1}^n (q_i)_s c_i = \left( \sum_{i=1}^n q_i c_i \right)_s.$$ 

   (Necessarily, $c_i = f(b_i)$, $i = 1, \ldots, n$.)

   We say that an ideal $I$ acts faithfully on $E$ if the annihilator $r_E I$ of $I$ in $E$ is zero. Since $E$ is faithful, then $B \to \text{End}_E B$ canonically, so an ideal $I$ acts faithfully iff \( \bigcap_{b \in I} \ker b = 0 \) (i.e. when each $b \in B$ is considered as an endomorphism of $E$). In this case we say $I$ has zero kernel in $E$. This implies that $I$ is a faithful ideal of $B$ inasmuch as $Ic = 0$ for some $c \in B$ implies $IcE = 0$, and then
   
   $$r_E I = 0 \Rightarrow cE = 0 \Rightarrow c = 0.$$ 

2. **FPF Theorem for Split-Null Extensions.** Let $E$ be a faithful $B$-module. Then, $R = (B,E)$ is an FPF ring iff the following three conditions hold:

   (2.1) $E$ is injective,
   
   (2.2) $BS^{-1} \cong \text{End}_B E$ canonically, where $S = \{ b \in B | ker b = 0 \}$,
   
   (2.3) Every finitely generated ideal of $B$ with zero kernel in $E$ is invertible in $BS^{-1}$.

   When this is so, then $Q_c(R) = (BS^{-1},E)$, is self-injective, and $I$ is a projective ideal of $B$.

3. **Corollary.** If $E$ is a strongly balanced injective module over $B$, then every finitely generated ideal of $B$ acting faithfully on $E$ is projective, so $B$ is pre-FPF.

4. **Corollary.** If there exists a strongly balanced injective torsion free module $E$ over a domain $B$, then $B$ is FPF, hence Prufer.

   **Comments.** Corollary 3 follows from the theorem, and the characterization in (IN 1) of injective $(B,E)$. Moreover, in Corollary 4, every finitely generated ideal
$I \neq 0$ acts faithfully on $E$, hence is projective by the theorem, so $B$ is pre-FPF. Since pre-FPF $\Rightarrow$ FPF $\Leftrightarrow$ Prüfer in a domain (see [2]), Corollary 4 follows.

Preliminaries.

Proof of Theorem 2. We first compute $Q_c(R)$. Let $(b,x) = (\frac{b}{x}b)$ denote a typical element of
\[
R = (B,E) = \left\{ \left( \begin{array}{c} b \\ x \\ b \end{array} \right) \in \left( \begin{array}{c} BE \\ OB \end{array} \right) \left| b \in B, x \in E \right. \right\}.
\]
If also $(c,y) \in R$, then
\[
(1) \quad (b,x)(c,y) = (bc, cx + by)
\]
so $(b,x)$ is regular in $R$ iff
\[
\ker b = \{ y \in E | by = 0 \} = 0.
\]
This follows since if $by = 0$ and $y \neq 0$, then $(b,x)(0,y) = (0,by) = 0$, so $(b,x)$ is not regular in $R$. Conversely, if we put (1) to 0, and $(c,y) \neq 0$, then $bc = 0$, so $E$ faithful implies $cE \neq 0$, and $0 \neq cE \subseteq \ker b$.

It follows that $S = (b \in B | \ker b = 0)$ is a multiplicative subset of $B^*$, so $Q = BS^{-1} \subseteq Q_c$. Furthermore,
\[
(2) \quad Q_c(R) \subseteq (Q, Q \otimes_B E) = (BS^{-1}, ES^{-1})
\]
since if $(b,x) \in R^*$, then
\[
(b,x)^{-1} = (b^{-1}, -b^{-1}x) \in (BS^{-1}, ES^{-1}) = (Q, Q \otimes_B E)
\]
where $b^{-1}x$ is identified with $b^{-1} \otimes x \in Q \otimes_B E$. Moreover, every
\[
q = (ab^{-1}, t^{-1}y) \in (BS^{-1}, ES^{-1})
\]
has the form
\[
q = (ab^{-1}, by)(b^{-1}t^{-1}, 0) \in BS^{-1}
\]
that is, $q \in Q_c(R)$, proving the inclusion (2) is an equality.

An ideal $K$ of $R$ has the form $K = (K_B, K_E)$, for an ideal $K_B \subseteq B$ and a $B$-submodule $K_E$ of $E$ such that $K_B E \subseteq K_E$. Since $(b,x)(c,y) = (bc, by + cx)$ $\forall b \in B, x, y \in E$, then $K$ is faithful in $R$ iff $K_B$ acts faithfully on $E$.

An ideal $K$ of $R$ generated by $\langle (a_\lambda, x_\lambda) \rangle_{\lambda \in \Lambda}$ has the form
\[
(5.1) \quad K = (K_B, K_E) = \sum_{\lambda \in \Lambda} (a_\lambda, x_\lambda) R = \left( \sum_{\lambda \in \Lambda} a_\lambda B, \sum_{\lambda \in \Lambda} a_\lambda E + \sum_{\lambda \in \Lambda} Bx_\lambda \right)
\]
\[
= (K_B, K_B E + \sum_{\lambda \in \Lambda} Bx_\lambda) = (K_B, K_B E + K_E).
\]

5.2. Proposition. An ideal $K$ (generated as in (5.1)) inverts in $Q_c(R) = (BS^{-1}, ES^{-1})$
iff $K_B$ inverts in $BS^{-1}$ and $x_\lambda \in K_B E$ for all $\lambda \in \Lambda$. In other words, $K$ is invertible in $R$ iff $K_B$ is invertible in $BS^{-1}$ and $K = (K_B, K_B E)$.
Proof. $K$ inverts in $(BS^{-1}, ES^{-1})$ iff there exist elements $q_1, \ldots, q_m \in BS^{-1}$ and $y_1, \ldots, y_m \in E$, $m < \infty$, such that $\sum_{i=1}^{m} K(q_i, y_i) = R$, equivalently
\[
\sum_{i=1}^{m} q_i K_B = B
\]
and
\[
E = \sum_{i=1}^{m} y_i K_B + \sum_{i=1}^{m} q_i K_E.
\]
Furthermore, (3) holds iff
\[
K_B \text{ inverts in } BS^{-1} \text{ and } K_B^{-1} = \sum_{i=1}^{m} q_i B.
\]
When (5) holds, then (4) is equivalent to
\[
E = \sum_{i=1}^{m} y_i K_B + K_B^{-1} K_E.
\]
Now by (5.1) (which is a straightforward computation), we have
\[
K_E = K_B E + \sum_{\lambda \in \Lambda} Bx_{\lambda}
\]
and so (6) is equivalent to
\[
E \supseteq \sum_{\lambda \in \Lambda} K_B^{-1} x_{\lambda}
\]
or
\[
x_{\lambda} \in K_B E \quad \forall \lambda \in \Lambda.
\]
Then, by (5.1), $K = (K_B, K_B E)$.

6. Corollary. If $E = Es$ for all $s \in S$, then $K$ inverts in $Q_c(R)$ iff $K_B$ inverts in $BS^{-1}$.

Proof. The necessity follows from the proposition. Conversely, if $K_B$ inverts in $BS^{-1}$, then $K_B$ contains an element $s \in S$, so then $E = Es$ implies that $E = K_B E$ and so the criterion of the proposition applies.

We can now complete the proof of Theorem 2.

Necessity. If $R = (B, E)$ is FPF, then by (FPF 1), $Q_c(R) = (BS^{-1}, ES^{-1})$ is self-injective. By (IN 1), then $ES^{-1}$ is a strongly balanced injective $BS^{-1}$-module. By [1], $R$ is integrally closed in $Q_c(R)$, hence contains all nilpotents of $(BS^{-1}, ES^{-1})$, so $R \supseteq (0, ES^{-1})$, that is,
\[
E = ES^{-1}.
\]
Since $E$ is thereby injective over $BS^{-1}$, then (2.1) holds via flatness of $BS^{-1}$ over $B$. Since $ES^{-1}$ is strongly balanced over $BS^{-1}$, then (2.2) holds.

Finally, if $I = \sum_{i=1}^{n} b_i B$ acts faithfully on $E$, and $F = \sum_{i=1}^{n} b_i E$, then
\[
K = (I, F) = \sum_{i=1}^{n} (b_i, 0) R
\]
is a finitely generated ideal of \( R \) and faithful, so \( R \) is FPF implies via Proposition 1 that \( K \) inverts in \( Q_c(R) \), hence \( I = K_B \) inverts in \( BS^{-1} \) by Proposition 5. This yields (2.3).

**Sufficiency.** (2.1) implies that \( E \) is divisible, hence \( E \) is canonically a \( BS^{-1} \)-module. Since \( BS^{-1} \) is a ring epic of \( B \), then \( E \) is also injective over \( BS^{-1} \). Now (2.2) evidently implies \( BS^{-1} = \text{End}_{BS^{-1}} E \) canonically, so \( Q_c(R) = (BS^{-1}, ES^{-1}) \) is self-injective by (IN 1), so \( R \) is (FPF 1).

It remains to show that \( R \) is (FPF 2). Let \( K = (K_B, K_E) \) be a finitely generated faithful ideal of \( R \). Then \( K_B \) acts faithfully on \( E \), as shown supra Proposition 5. Since \( K_B \) is finitely generated in \( B \), then \( K_B \) is invertible by (2.3), so \( K \) is invertible by Corollary 6.

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**Problems.** 1. Characterize when \( R = (B, E) \) is FPF for a general faithful bimodule \( E \) over a noncommutative ring. Unfortunately, arbitrary noncommutative FPF rings have yet to be characterized, although the list of characterized FPF rings includes (1) semiprime, (2) prime, (3) self-injective, (4) Noetherian semiprime, (5) semiperfect Noetherian etc. (see [6]).

2. Characterize \( FP^2 \) split-null extensions. Even for commutative \( R \) this is open, essentially since commutative \( FP^2 \) rings have not been characterized: Here is a "working" conjecture: A commutative ring \( R \) is \( FP^2 \) iff \( R \) is pre-\( FP^2 \) (i.e. finitely presented faithful ideals are projective) and \( Q_c(R) \) is \( FP \)-injective. The sufficiency of these conditions follows as in the proof of the corresponding conditions (FPF 1 and 2) in [1]. (Hint: an \( FP \)-injective ring splits in any finitely-presented over-module.)

3. If \( B = \text{End}_B E \) and \( E \) is injective over \( B \), is \( B \) necessarily FPF? The answer is yes when \( B \) is of Noetherian domain, since then the problem may be reduced to the case where \( E \) is indecomposable, hence then \( B = \text{End}_B E \) is the ring of \( p \)-adics (where \( P \) is the prime in \( \text{Ass} E \); see [3] and [5], Chapter 11). Thus, in this case \( B \) is a discrete valuation domain hence a PID, so a fortiori FPF.

**References**


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