PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 90, Number 2, February 1984

COMMUTATIVE FPF RINGS ARISING AS SPLIT-NULL EXTENSIONS

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ABSTRACT. Let $R = (B,E)$ be the split-null or trivial extension of a faithful module E over a commutative ring B. R is an FPF ring iff the partial quotient ring BS^{-1} with respect to the set S of elements of B with zero annihilator in E is canonically the endomorphism ring of E, that is $BS^{-1} = \text{End}_B ES^{-1}$, every finitely generated ideal with zero annihilator in E is invertible in BS^{-1}, and $E = ES^{-1}$ is an injective module over B. The proof uses the author's characterization of commutative FPF rings [1] and also the characterization of self-injectivity of a split-null extension [3].

Background. A ring R is (*right*) [F]PF if every [finitely generated] faithful right R-module generates the category mod-R of all right R-modules. A module M is (Beachy-Blair) *cofaithful* if there is an embedding $R \to M^n$ for some finite integer n. Over a commutative ring R, every finitely generated faithful module is manifestly cofaithful (= CF-faithful in [4]). Furthermore, any cofaithful module over a right self-injective ring R generates mod-R, since then R splits in any over module, hence $M^n \approx R \oplus X$ in mod-R for some $n > 0$. This shows that any commutative self-injective ring R is FPF. Moreover, a commutative FPF ring R is characterized in [1] by the two conditions:

(FPF 1) R is quotient-injective, i.e. the (classical) quotient ring $Q_c(R)$ is injective.

(FPF 2) R is pre-FPF, i.e. finitely generated faithful ideals are generators of mod-R (equivalently, are finitely generated projective [1, 2]).

Since any commutative self-injective ring is FPF, it is thereby pre-FPF, that is, $(\text{FPF}_1) \Rightarrow (\text{FPF}_2)$ for injective R: in fact, then R is the only finitely generated faithful ideal.

In [3] we considered the *split-null* (or trivial) extension $R = (B,E)$ of a faithful B-bimodule E over a ring B, and characterized the conditions under which (B,E) is right PF (resp. right self-injective). We say that E is *left strongly balanced* [3] if B is canonically isomorphic to the endomorphism ring of the *right* B-module E; notation: $B = \text{End}\, E_B$. Theorem 2 of [3] states

(IN 1) (B,E) is right self-injective iff E is a left strongly balanced injective right B-module.

(IN 2) (B,E) is right PF iff E is a left strongly balanced injective right cogenerator over B.

Henceforth let B be a commutative ring, and let E be a faithful B-module. Then $R = (B,E)$ is commutative, and our main theorem characterizes when R is FPF. To

Received by the editors September 27, 1982.

1980 *Mathematics Subject Classification*. Primary 16A36, 16A52; Secondary 16A14.

describe the result, let S denote the multiplicative set of B consisting of all $b \in B$ such that $\ker b = 0$, i.e., $b\colon E \to E$ is monic. Then, S is a subset of the set B^* of regular elements of B, hence the quotient ring $Q = BS^{-1}$ embeds in $Q = Q_c(B)$ canonically.

1. PROPOSITION AND DEFINITION. *An ideal I of B is said to invert or is invertible in a commutative overring Q provided the equivalent conditions hold*:

(a) $I'I = B$ *for* $I' = (I:B) = \{q \in Q | qI \subseteq B\}$.

(b) $\sum_{i=1}^n q_i I = B$ *for finitely many elements* $q_1, \dots, q_n \in Q$.

(c) *There exist elements* $q_i, \dots, q_n \in Q$, *and* $b_i, \dots, b_n \in I$ *so that* $q_i b_i \in B$, $i = 1, \dots, n$ *and* $\sum_{i=1}^n q_i b_i = 1$.

(d) *I is a faithful ideal of B, and there exist* $q_i, \dots, q_n \in Q$, $b_i, \dots, b_n \in I$ *so that* $q_i b_i \in B_i$, $i = 1, \dots, n$ *and* $x = \sum_{i=1}^n b_i q_i(x)\ \forall x \in I$.

(e) *I is a finitely generated faithful projective B-module, and if* $f \in \mathrm{Hom}_B(I,B)$, *then* $f = q_s$, *for some* $q \in Q$, *where* $q_s(x) = qx\ \forall x \in B$.

(f) *I generates* mod-B *and if* $f \in \mathrm{Hom}_B(I,B)$, *then* $f = q_s$ *for some* $q \in Q$.

PROOF. (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) is direct, and (d) $\Leftrightarrow$ (e) uses the dual basis lemma. It can be shown that $b_1, \dots, b_n$ generates I, that $(q_1)_s, \dots, (q_n)_s$ generates $\mathrm{Hom}_B(I,B)$, and if $f \in \mathrm{Hom}_B(I,B)$, then there exists $c_i \in B$, $i = 1, \dots, n$, so that

$$f = \sum_{i=1}^n (q_i)_s c_i = \left(\sum_{i=1}^n q_i c_i \right)_s .$$

(Necessarily, $c_i = f(b_i)$, $i = 1, \dots, n$.)

We say that an ideal I acts *faithfully on E* if the annihilator $r_E I$ of I in E is zero. Since E is faithful, then $B \to \mathrm{End}\, E_B$ canonically, so an ideal I acts faithfully iff $\bigcap_{b \in I} \ker b = 0$ (i.e. when each $b \in B$ is considered as an endomorphism of E). In this case we say I has *zero kernel* in E. This implies that I is a faithful ideal of B inasmuch as $Ic = 0$ for some $c \in B$ implies $IcE = 0$, and then

$$r_E I = 0 \Rightarrow cE = 0 \Rightarrow c = 0.$$

2. FPF THEOREM FOR SPLIT-NULL EXTENSIONS. *Let E be a faithful B-module. Then, $R = (B,E)$ is an* FPF *ring iff the following three conditions hold*:

(2.1) *E is injective*,

(2.2) $BS^{-1} \approx \mathrm{End}_B E$ *canonically, where* $S = \{b \in B | \ker b = 0\}$,

(2.3) *Every finitely generated ideal of B with zero kernel in E is invertible in* BS^{-1}.

When this is so, then $Q_c(R) = (BS^{-1}, E)$, *is self-injective, and I is a projective ideal of B.*

3. COROLLARY. *If E is a strongly balanced injective module over B, then every finitely generated ideal of B acting faithfully on E is projective, so B is pre-*FPF.

4. COROLLARY. *If there exists a strongly balanced injective torsion free module E over a domain B, then B is* FPF, *hence Prufer.*

Comments. Corollary 3 follows from the theorem, and the characterization in (IN 1) of injective (B,E). Moreover, in Corollary 4, every finitely generated ideal

$I \neq 0$ acts faithfully on E, hence is projective by the theorem, so B is pre-FPF. Since pre-FPF $\Rightarrow$ FPF $\Leftrightarrow$ Prufer in a domain (see [**2**]), Corollary 4 follows.

Preliminaries.

PROOF OF THEOREM 2. We first compute $Q_c(R)$. Let $(b,x) = \left(\begin{smallmatrix} b & x \\ o & b\end{smallmatrix}\right)$ denote a typical element of

$$R = (B,E) = \left\{ \begin{pmatrix} b & x \\ o & b \end{pmatrix} \in \begin{pmatrix} BE \\ OB \end{pmatrix} \middle| b \in B, x \in E \right\}.$$

If also $(c,y) \in R$, then

$$(1) \qquad (b,x)(c,y) = (bc, cx + by)$$

so (b,x) is regular in R iff

$$\ker b = \{ y \in E | by = 0 \} = 0.$$

This follows since if $by = 0$ and $y \neq 0$, then $(b,x)(0,y) = (0,by) = 0$, so (b,x) is not regular in R. Conversely, if we put (1) to 0, and $(c,y) \neq 0$, then $bc = 0$, so E faithful implies $cE \neq 0$, and $0 \neq cE \subseteq \ker b$.

It follows that $S = \{b \in B | \ker b = 0\}$ is a multiplicative subset of B^*, so $Q = BS^{-1} \subseteq Q_c$. Furthermore,

$$(2) \qquad Q_c(R) \subseteq (Q, Q \otimes_B E) = (BS^{-1}, ES^{-1})$$

since if $(b,x) \in R^*$, then

$$(b,x)^{-1} = (b^{-1}, -b^{-1}x) \in (BS^{-1}, ES^{-1}) = (Q, Q \otimes_B E)$$

where $b^{-1}x$ is identified with $b^{-1} \otimes x \in Q \otimes_B E$. Moreover, every

$$q = (ab^{-1}, t^{-1}y) \in (BS^{-1}, ES^{-1})$$

has the form

$$q = (at, by)(b^{-1}t^{-1}, 0) \in BS^{-1}$$

that is, $q \in Q_c(R)$, proving the inclusion (2) is an equality.

An ideal K of R has the form $K = (K_B, K_E)$, for an ideal $K_B \subseteq B$ and a B-submodule K_E of E such that $K_B E \subseteq K_E$. Since $(b,x)(c,y) = (bc, by + cx)$ $\forall b \in B$, $x,y \in E$, then K is faithful in R iff K_B acts faithfully on E.

An ideal K of R generated by $\{(a_\lambda, x_\lambda)\}_{\lambda \in \Lambda}$ has the form

$$(5.1) \quad K = (K_B, K_E) = \sum_{\lambda \in \Lambda} (a_\lambda, x_\lambda) R = \left(\sum_{\lambda \in \Lambda} a_\lambda B, \sum_{\lambda \in \Lambda} a_\lambda E + \sum_{\lambda \in \Lambda} B x_\lambda \right)$$
$$= \left(K_B, K_B E + \sum_{\lambda \in \Lambda} B x_\lambda \right) = (K_B, K_B E + K_E).$$

5.2. PROPOSITION. *An ideal K (generated as in* (5.1)) *inverts in*

$$Q_c(R) = (BS^{-1}, ES^{-1})$$

iff K_B inverts in BS^{-1} and $x_\lambda \in K_B E$ for all $\lambda \in \Lambda$. In other words, K is invertible in R iff K_B is invertible in BS^{-1} and $K = (K_B, K_B E)$.

PROOF. K inverts in (BS^{-1}, ES^{-1}) iff there exist elements $q_1, \ldots, q_m \in BS^{-1}$ and $y_1, \ldots, y_m \in E$, $m < \infty$, such that $\sum_{i=1}^n K(q_i, y_i) = R$, equivalently

(3)
$$\sum_{i=1}^{n} q_i K_B = B$$

and

(4)
$$E = \sum_{i=1}^{m} y_i K_B + \sum_{i=1}^{m} q_i K_E.$$

Furthermore, (3) holds iff

(5)
$$K_B \text{ inverts in } BS^{-1} \quad \text{and} \quad K_B^{-1} = \sum_{i=1}^{m} q_i B.$$

When (5) holds, then (4) is equivalent to

(6)
$$E = \sum_{i=1}^{m} y_i K_B + K_B^{-1} K_E.$$

Now by (5.1) (which is a straightforward computation), we have

(7)
$$K_E = K_B E + \sum_{\lambda \in \Lambda} B x_\lambda$$

and so (6) is equivalent to

(8)
$$E \supseteq \sum_{\lambda \in \Lambda} K_B^{-1} x_\lambda$$

or

(9)
$$x_\lambda \in K_B E \quad \forall \lambda \in \Lambda.$$

Then, by (5.1), $K = (K_B, K_B E)$.

6. COROLLARY. *If $E = Es$ for all $s \in S$, then K inverts in $Q_c(R)$ iff K_B inverts in BS^{-1}.*

PROOF. The necessity follows from the proposition. Conversely, if K_B inverts in BS^{-1}, then K_B contains an element $s \in S$, so then $E = Es$ implies that $E = K_B E$ and so the criterion of the proposition applies.

We can now complete the proof of Theorem 2.

Necessity. If $R = (B, E)$ is FPF, then by (FPF 1), $Q_c(R) = (BS^{-1}, ES^{-1})$ is self-injective. By (IN 1), then ES^{-1} is a strongly balanced injective BS^{-1}-module. By [**1**], R is integrally closed in $Q_c(R)$, hence contains all nilpotents of (BS^{-1}, ES^{-1}), so $R \supseteq (0, ES^{-1})$, that is,

(10)
$$E = ES^{-1}.$$

Since E is thereby injective over BS^{-1}, then (2.1) holds via flatness of BS^{-1} over B. Since ES^{-1} is strongly balanced over BS^{-1}, then (2.2) holds.

Finally, if $I = \sum_{i=1} b_i B$ acts faithfully on E, and $F = \sum_{i=1}^n b_i E$, then

$$K = (I, F) = \sum_{i=1}^{n} (b_i, 0) R$$

is a finitely generated ideal of R and faithful, so R FPF implies via Proposition 1 that K inverts in $Q_c(R)$, hence $I = K_B$ inverts in BS^{-1} by Proposition 5. This yields (2.3).

Sufficiency. (2.1) implies that E is divisible, hence E is canonically a BS^{-1}-module. Since BS^{-1} is a ring epic of B, then E is also injective over BS^{-1}. Now (2.2) evidently implies $BS^{-1} = \mathrm{End}_{BS^{-1}} E$ canonically, so $Q_c(R) = (BS^{-1}, ES^{-1})$ is self-injective by (IN 1), so R is (FPF 1).

It remains to show that R is (FPF 2). Let $K = (K_B, K_E)$ be a finitely generated faithful ideal of R. Then K_B acts faithfully on E, as shown *supra* Proposition 5. Since K_B is finitely generated in B, then K_B is invertible by (2.3), so K is invertible by Corollary 6.

ACKNOWLEDGEMENT. I wish to acknowledge the referee for suggestions that improved the exposition.

Problems. 1. Characterize when $R = (B, E)$ is FPF for a general faithful bimodule E over a noncommutative ring. Unfortunately, arbitrary noncommutative FPF rings have yet to be characterized, although the list of characterized FPF rings includes (1) semiprime, (2) prime, (3) self-injective, (4) Noetherian semiprime, (5) semiperfect Noetherian etc. (see **[6]**).

2. Characterize $\mathrm{FP}^2\mathrm{F}$ split-null extensions. Even for commutative R this is open, essentially since commutative $\mathrm{FP}^2\mathrm{F}$ rings have not been characterized: Here is a "working" conjecture: A commutative ring R is $\mathrm{FP}^2\mathrm{F}$ iff R is pre-$\mathrm{FP}^2\mathrm{F}$ (i.e. finitely presented faithful ideals are projective) and $Q_c(R)$ is FP-injective. The sufficiency of these conditions follows as in the proof of the corresponding conditions (FPF 1 and 2) in **[1]**. (Hint: an FP-injective ring splits in any finitely-presented over-module.)

3. If $B = \mathrm{End}_B E$ and E is injective over B, is B necessarily FPF? The answer is yes when B is of Noetherian domain, since then the problem may be reduced to the case where E is indecomposable, hence then $B = \mathrm{End}_B E$ is the ring of p-adics (where P is the prime in $\mathrm{Ass}\,E$; see **[3]** and **[5]**, Chapter 11). Thus, in this case B is a discrete valuation domain hence a PID, so *a fortiori* FPF.

REFERENCES

1. C. Faith, *Injective quotient rings of commutative rings*. II, Injective Modules and Injective Quotient Rings, Lecture Notes in Pure and Appl. Math., vol. 72, Dekker, New York, 1982.

2. ______, *Injective quotient rings of commutative rings*. I, Module Theory, Lecture Notes in Math., vol. 700 (C. Faith and S. Wiegand, eds.), Springer-Verlag, Berlin and New York, 1979.

3. ______, *Self-injective rings*, Proc. Amer. Math. Soc. **77** (1979), 157–164.

4. ______, *Algebra* II: *Ring theory*, Springer-Verlag, Berlin and New York, 1976.

5. ______, *Injective modules over Levitzki rings*, Injective Modules and Injective Quotient Rings, Lecture Notes in Pure and Appl. Math., vol. 72, Dekker, New York, 1982.

6. C. Faith and S. Page, *FPF ring theory: faithful modules and generators of* Mod-R, Lecture Notes of the London Math. Soc., Cambridge Univ. Press, 1984.

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