PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 90, Number 2, February 1984

MORITA EQUIVALENCE AND INFINITE MATRIX RINGS

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ABSTRACT. This paper contains the proof of a theorem conjectured by William Stephenson in his thesis [3].

Two rings R and S are Morita equivalent if their categories of right R modules are isomorphic. A standard example of two Morita equivalent rings are R and R_n, the $n \times n$ matrices over R. It is therefore true that if R_n and S_m are isomorphic for integers n and m then R and S are Morita equivalent. An easy example (below) shows that there are Morita equivalent rings R and S with R_n and S_m not isomorphic for any integers m and n. This means that category isomorphism cannot be reduced to a ring isomorphism in this way. Regard R_n as End $R^{(n)}$, where $R^{(n)}$ is a direct sum of n copies of R. Let $R^{(N)}$ be a direct sum of countably many copies of R. It follows from a result of Eilenberg (below) that if R and S are Morita equivalent, then End $R^{(N)} \approx$ End $S^{(N)}$. We prove the following

THEOREM. *R and S are Morita equivalent if and only if* End $R^{(N)} \approx$ End $S^{(N)}$, *where N is a countably infinite set.*

Note End $R^{(N)} = R_N$ is just the ring of column finite matrices over R.

The following example was suggested by Kent Fuller. I would also like to thank him for several helpful conversations.

EXAMPLE. *Two Morita equivalent rings R and S with R_n not isomorphic to S_m for any integers n and m.*

Let K be any field, $R \approx K \times K$ and $S \approx K \times K_2$. Then $R_n \approx K_n \times K_n$ and $S_m \approx K_m \times K_{2m}$. By the uniqueness part of the Wedderburn theorem, the latter two rings cannot be isomorphic, but R and S are clearly Morita equivalent.

Our result is motivated by, and requires, a result of Eilenberg. This is an exercise in Anderson and Fuller [1, p. 202]. Let R be a ring. Recall that P is a progenerator if P is a finitely generated projective generator, that is, $R^{(n)} \approx P \oplus K$ and $P^{(m)} \approx R \oplus L$ for modules K and L, and integers m and n.

NOTATION. $X^{(N)}$ always denotes the direct sum of a countable number of copies of X.

For completeness we sketch a proof of the following

PROPOSITION (EILENBERG). *If P is a progenerator*, $P^{(N)} \approx R^{(N)}$.

We use the notation preceding the proposition, and note that without loss of generality $n = m$. Only one substitution is made on each line.

Received by the editors January 25, 1983 and, in revised form, June 23, 1983.
1980 *Mathematics Subject Classification*. Primary 16A89, 16A49; Secondary 16A36, 16A42.

We have

$$P^{(N)} \approx R^{(N)} \oplus L^{(N)} \approx R^{(N)} \oplus R^{(N)} \oplus L^{(N)} \approx R^{(N)} \oplus P^{(N)}.$$

Similarly, $R^{(N)} \approx P^{(N)} \oplus R^{(N)}$, and we are done.

To obtain our results we will need an observation about projective modules. If X is any module and $\{Y_i\}$ is an infinite set of modules, then $G = \operatorname{Hom}(X, \Sigma \oplus Y_i)$ may clearly be described as the subgroup of $\Pi \operatorname{Hom}(X, Y_i)$ with the property that $f = (f_1, \ldots, f_n, \ldots)$ is in G if and only if for every $x \in X$, $f_i(x) = 0$ for all but finitely many i. We thus have a canonical embedding

$$c\colon \sum \oplus \operatorname{Hom}(X, Y_i) \to \operatorname{Hom}\left(X, \sum \oplus Y_i\right).$$

If X is finitely generated then c is obviously an isomorphism. For projective modules we observe:

LEMMA 1. *Let P_R be a projective module. Then P_R is finitely generated if and only if the c obtained from the map above by letting $X = P$ and $Y_i = P$ is an isomorphism.*

PROOF. Recall that P is projective if and only if P has a dual basis, i.e., a set $\{(p_i, f_i)\}$ where $p_i \in P$ and $f_i \in \operatorname{Hom}(P, R)$, such that for a given $p \in P$:

1. $f_i(p) = 0$ for almost all i and
2. $p = \Sigma p_i f_i(p)$.

Now if c is an isomorphism, the map $g\colon P \to P^{(N)}$ given by $g(p) = (p_1 f_1(p), \ldots, p_n f_n(p), \ldots)$ must be identically zero from some point on, so clearly P is finitely generated. As observed in the remark preceding the lemma, the converse is obvious.

Before we prove our theorem we recall the Morita theorem.

MORITA THEOREM. *Let R and S be rings. Then R and S have isomorphic categories of right modules if and only if there is a finitely generated projective generator P_R such that $S = \operatorname{End} P_R$. The category isomorphism is given by $X_R \to \operatorname{Hom}(P_R, X_R)$.*

The functor $\operatorname{Hom}(P_R, -)$ does the usual on maps.

Before we prove our theorem, let us take a look at Eilenberg's proposition. If we take the endomorphism ring of both sides of the isomorphism in the proposition, we obtain $R_N \approx S_N$, where $S = \operatorname{End} P$. Note, $\operatorname{End} X^{(N)} \approx (\operatorname{End} X)_N$ in general only when X is finitely generated. Thus, by Morita's theorem, if R and S are Morita equivalent, $R_N \approx S_N$. This was the motivation for our theorem. What we need to do is, given an isomorphism between the column finite matrices, pick out a progenerator that works.

PROOF OF THE THEOREM. Notation: $U = R^{(N)} = \Sigma \oplus u_i R$; $V = S^{(N)} = \Sigma \oplus v_i S$; $\{e_{ij}\}$ are the usual matrix units for U, i.e., $e_{ij}(u_j) = u_i$ and $e_{ij}(u_k) = 0$ if $k \neq j$. $\{f_{ij}\}$ are the corresponding matrix units for V. Note that in this case, Σe_{ii} and Σf_{ii} makes not sense. Finally, σ is an isomorphism from R_N to S_N.

To establish our result we need to find and S progenerator P with $R = \operatorname{End} P$. There is an obvious candidate; $\sigma(e_{11})$ is an idempotent in $\operatorname{End} V$, therefore $V = \sigma(e_{11})V \oplus C$, where C is a complement. $\sigma(e_{11})V$, being a summand of a free module, is projective, but not obviously finitely generated nor a generator. We prove this.

$\sigma(e_{11})V$ is finitely generated. We first note that $\sigma(e_{ii})V \approx \sigma(e_{11})V$ for all i. This is just a standard calculation with matrix units. Let $h_{ij} = \sigma(f_{ij})$;

$$h_{ii}V \supset h_{ii}h_{ij}V = h_{ij}V \supset h_{ij}h_{ji}V = h_{ii}V,$$

so $h_{ii}V = h_{ij}V$. So, $h_{ji}(h_{ii}V) \subset h_{jj}V$ and $h_{ij}(h_{jj}V) \supset h_{ii}V$, and the composition of the two maps given by e_{ji} and e_{ij} is the identity on $e_{ii}V$.

We therefore have that $\sigma(e_{11})V \approx \sigma(e_{ii})V$ for all i, and note that $\Sigma\sigma(e_{ii})V$ is direct. Now, if $\sigma(e_{11})V$ is not finitely generated, we may construct a map h: $\sigma(e_{11})V \to \Sigma\sigma(e_{ii}V)$ as in the lemma, using a dual basis for $\sigma(e_{11})V$. h has the property that $\operatorname{Im} h$ is not contained in any finite sum of the $\sigma(e_{ii})V$. Now, since $\sigma(e_{11})V$ is a summand for V we can extend the domain of h to all of V by making it zero on the complement, which we still call h.

Now, go back to U, and look at $\sigma^{-1}(h)e_{11}$. We know that $\sigma^{-1}(h)e_{11}u_i = 0$ for $i \neq 1$, so $\sigma^{-1}(h)e_{11}U = (\sigma^{-1}(h)e_{11}u_1)R$, and this latter module is principal, therefore it is contained in a finite sum $\Sigma_{i=1}^{K} u_iR$. Therefore, $e_{11}\sigma^{-1}he_{11} = 0$ for all $i > K$, so $\sigma(e_{ii})\sigma(\sigma^{-1}(h))\sigma(e_{11}) = 0$ for all $i > k$, i.e., $\sigma(e_{ii})h\sigma(e_{11}) = 0$ for all $i > K$, a contradiction, which shows that $\sigma(e_{11})V$ is finitely generated.

$\sigma(e_{11})V$ is a generator. Look at $\sigma^{-1}(f_{11})$. By the previous argument, $\sigma^{-1}(f_{11})U$ is finitely generated, and so is contained in a finite sum of the u_iR. Thus there is a K with

$$(e_{11} + \cdots + e_{KK})\sigma^{-1}(f_{11}) = \sigma^{-1}(f_{11}).$$

So, applying σ, we have $(\sigma(e_{11}) + \cdots + \sigma(e_{KK}))f_{11} = f_{11}$. Multiply on the left to get

$$[f_{11}\sigma(e_{11}) + \cdots + f_{11}\sigma(e_{KK})]f_{11} = f_{11}.$$

Now evaluate at v_1 to get

$$[f_{11}\sigma(e_{11}) + \cdots + f_{11}\sigma(e_{KK})]v_1 = v_1. \tag{1}$$

Now let $G = \sigma(e_{11})V \oplus \cdots \oplus \sigma(e_{KK})V$ (external direct sum), and consider the map: f: $G \to v_1S$ given by

$$(\sigma(e_{11})v_1, \ldots, \sigma(e_{KK})v_K) \to f_{11}(\sigma(e_{11})v_1 + \cdots + \sigma(e_{KK})v_K).$$

Then (1) shows that f is an epimorphism. Since $v_1S \approx S$, f splits, and since $\sigma(e_{ii})V \approx \sigma(e_{11})V$, we have $[\sigma(e_{11})V]^{(K)} \approx S \oplus C$, so $\sigma(e_{11})V$ is a generator.

End $\sigma(e_{11})V \approx R$. Let M be any module and $M = N \oplus C$. Let e be the projection onto M along C. Then it is well known and easy to prove that End $N \approx e(\text{End } M)e$, so since $V = \sigma(e_{11})(V) \oplus C$, End $\sigma(e_{11})V = \sigma(e_{11})(\text{End } V)\sigma(e_{11})$. Apply σ^{-1}:

$$\text{End } \sigma(e_{11})V \approx e_{11}\sigma^{-1}(\text{End } V)e_{11} = e_{11}R_Ne_{11} \approx R.$$

References

1. F. Anderson and K. Fuller, *Rings and categories of modules*, Springer-Verlag, New York, 1973.

2. K. Morita, *Duality of modules and its applications to the theory of rings with minimum condition*, Sci. Rep. Tokyo Kyoiku Daigaku Sect. A **6** (1958), 85–142.

3. W. Stephenson, Ph. D. Thesis, Univ. of London, 1967.

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