MORITA EQUIVALENCE AND INFINITE MATRIX RINGS

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Abstract. This paper contains the proof of a theorem conjectured by William Stephenson in his thesis [3].

Two rings $R$ and $S$ are Morita equivalent if their categories of right $R$ modules are isomorphic. A standard example of two Morita equivalent rings are $R$ and $R_n$, the $n \times n$ matrices over $R$. It is therefore true that if $R_n$ and $S_m$ are isomorphic for integers $n$ and $m$ then $R$ and $S$ are Morita equivalent. An easy example (below) shows that there are Morita equivalent rings $R$ and $S$ with $R_n$ and $S_m$ not isomorphic for any integers $m$ and $n$. This means that category isomorphism cannot be reduced to a ring isomorphism in this way. Regard $R_n$ as $\text{End } R^n$, where $R^n$ is a direct sum of $n$ copies of $R$. Let $R^{(N)}$ be a direct sum of countably many copies of $R$. It follows from a result of Eilenberg (below) that if $R$ and $S$ are Morita equivalent, then $\text{End } R^{(N)} \approx \text{End } S^{(N)}$.

We prove the following

Theorem. $R$ and $S$ are Morita equivalent if and only if $\text{End } R^{(N)} \approx \text{End } S^{(N)}$, where $N$ is a countably infinite set.

Note $\text{End } R^{(N)} = R_N$ is just the ring of column finite matrices over $R$.

The following example was suggested by Kent Fuller. I would also like to thank him for several helpful conversations.

Example. Two Morita equivalent rings $R$ and $S$ with $R_n$ not isomorphic to $S_m$ for any integers $n$ and $m$.

Let $K$ be any field, $R \approx K \times K$ and $S \approx K \times K_2$. Then $R_n \approx K_n \times K_n$ and $S_m \approx K_m \times K_{2m}$. By the uniqueness part of the Wedderburn theorem, the latter two rings cannot be isomorphic, but $R$ and $S$ are clearly Morita equivalent.

Our result is motivated by, and requires, a result of Eilenberg. This is an exercise in Anderson and Fuller [1, p. 202]. Let $R$ be a ring. Recall that $P$ is a progenerator if $P$ is a finitely generated projective generator, that is, $R^{(n)} \approx P \oplus K$ and $P^{(m)} \approx R \oplus L$ for modules $K$ and $L$, and integers $m$ and $n$.

Notation. $X^{(N)}$ always denotes the direct sum of a countable number of copies of $X$.

For completeness we sketch a proof of the following

Proposition (Eilenberg). If $P$ is a progenerator, $P^{(N)} \approx R^{(N)}$.

We use the notation preceding the proposition, and note that without loss of generality $n = m$. Only one substitution is made on each line.
We have
\[ P^{(N)} \approx R^{(N)} \oplus L^{(N)} \approx R^{(N)} \oplus R^{(N)} \oplus L^{(N)} \approx R^{(N)} \oplus P^{(N)}. \]
Similarly, \( R^{(N)} \approx P^{(N)} \oplus R^{(N)} \), and we are done.

To obtain our results we will need an observation about projective modules. If \( X \) is any module and \( \{Y_i\} \) is an infinite set of modules, then \( G = \text{Hom}(X, \sum \oplus Y_i) \) may clearly be described as the subgroup of \( \prod \text{Hom}(X, Y_i) \) with the property that \( f = (f_1, \ldots, f_n, \ldots) \) is in \( G \) if and only if for every \( x \in X, f_i(x) = 0 \) for all but finitely many \( i \). We thus have a canonical embedding
\[ c: \sum \oplus \text{Hom}(X, Y_i) \rightarrow \text{Hom}(X, \sum \oplus Y_i). \]
If \( X \) is finitely generated then \( c \) is obviously an isomorphism. For projective modules we observe:

**Lemma 1.** Let \( P_R \) be a projective module. Then \( P_R \) is finitely generated if and only if the \( c \) obtained from the map above by letting \( X = P \) and \( Y_i = P \) is an isomorphism.

**Proof.** Recall that \( A \) is projective if and only if \( A \) has a dual basis, i.e., a set \( \{(p_i, f_i)\} \) where \( p_i \in P \) and \( f_i \in \text{Hom}(P, R) \), such that for a given \( p \in P \):

1. \( f_i(p) = 0 \) for almost all \( i \)
2. \( p = \sum p_i f_i(p) \).

Now if \( c \) is an isomorphism, the map \( g: P \rightarrow P^{(N)} \) given by \( g(p) = (p_1 f_1(p), \ldots, p_n f_n(p), \ldots) \) must be identically zero from some point on, so clearly \( P \) is finitely generated. As observed in the remark preceding the lemma, the converse is obvious.

Before we prove our theorem we recall the Morita theorem.

**Morita Theorem.** Let \( R \) and \( S \) be rings. Then \( R \) and \( S \) have isomorphic categories of right modules if and only if there is a finitely generated projective generator \( P_R \) such that \( S = \text{End} P_R \). The category isomorphism is given by \( X_R \rightarrow \text{Hom}(P_R, X_R) \).

The functor \( \text{Hom}(P_R, -) \) does the usual on maps.

Before we prove our theorem, let us take a look at Eilenberg’s proposition. If we take the endomorphism ring of both sides of the isomorphism in the proposition, we obtain \( R_N \approx S_N \), where \( S = \text{End} P \). Note, \( \text{End} X^{(N)} \approx (\text{End} X)_N \) in general only when \( X \) is finitely generated. Thus, by Morita’s theorem, if \( R \) and \( S \) are Morita equivalent, \( R_N \approx S_N \). This was the motivation for our theorem. What we need to do is, given an isomorphism between the column finite matrices, pick out a progenerator that works.

**Proof of The Theorem.** Notation: \( U = R^{(N)} = \sum \oplus u_i R; V = S^{(N)} = \sum \oplus v_i S; \{e_{ij}\} \) are the usual matrix units for \( U \), i.e., \( e_{ij}(u_j) = u_i \) and \( e_{ij}(u_k) = 0 \) if \( k \neq j \). \( \{f_{ij}\} \) are the corresponding matrix units for \( V \). Note that in this case, \( \Sigma e_{ii} \) and \( \Sigma f_{ii} \) makes not sense. Finally, \( \sigma \) is an isomorphism from \( R_N \) to \( S_N \).

To establish our result we need to find a \( S \) progenerator \( P \) with \( R = \text{End} P \). There is an obvious candidate; \( \sigma(e_{11}) \) is an idempotent in \( \text{End} V \), therefore \( V = \sigma(e_{11}) V \oplus C \), where \( C \) is a complement. \( \sigma(e_{11}) V \), being a summand of a free module, is projective, but not obviously finitely generated nor a generator. We prove this.
\( \sigma(e_{11})V \) is finitely generated. We first note that \( \sigma(e_{ii})V \approx \sigma(e_{11})V \) for all \( i \). This is just a standard calculation with matrix units. Let \( h_{ij} = \sigma(f_{ij}) \);

\[
h_{ii}V \supset h_{i1}h_{11}V = h_{11}V \supset h_{1i}h_{ij}V = h_{ii}V,
\]

so \( h_{ii}V = h_{ij}V \). So, \( h_{ij}(h_{ii}V) \subset h_{jj}V \) and \( h_{ij}(h_{jj}V) \supset h_{ii}V \), and the composition of the two maps given by \( e_{ii} \) and \( e_{ij} \) is the identity on \( e_{ii}V \).

We therefore have that \( \sigma(e_{11})V \approx \sigma(e_{ii})V \) for all \( i \), and note that \( \sum \sigma(e_{ii})V \) is direct. Now, if \( \sigma(e_{11})V \) is not finitely generated, we may construct a map \( h: \sigma(e_{11})V \to \sum \sigma(e_{ii})V \) as in the lemma, using a dual basis for \( \sigma(e_{11})V \). \( h \) has the property that \( \text{Im } h \) is not contained in any finite sum of the \( \sigma(e_{ii})V \). Now, since \( \sigma(e_{11})V \) is a summand for \( V \) we can extend the domain of \( h \) to all of \( V \) by making it zero on the complement, which we still call \( h \).

Now, go back to \( U \), and look at \( \sigma^{-1}(h)e_{11} \). We know that \( \sigma^{-1}(h)e_{11}u_i = 0 \) for \( i \neq 1 \), so \( \sigma^{-1}(h)e_{11}U = (\sigma^{-1}(h)e_{11}u_1)R \), and this latter module is principal, therefore it is contained in a finite sum \( \sum_{i=1}^K u_i R \). Therefore, \( e_{11}\sigma^{-1}(h)e_{11} = 0 \) for all \( i > K \), so \( \sigma(e_{ii})\sigma(\sigma^{-1}(h))\sigma(e_{11}) = 0 \) for all \( i > k \), i.e., \( \sigma(e_{ii})\sigma(e_{11}) = 0 \) for all \( i > K \), a contradiction, which shows that \( \sigma(e_{11})V \) is finitely generated.

\( \sigma(e_{11})V \) is a generator. Look at \( \sigma^{-1}(f_{11}) \). By the previous argument, \( \sigma^{-1}(f_{11})U \) is finitely generated, and so is contained in a finite sum of the \( u_i R \). Thus there is a \( K \) with

\[
(e_{11} + \cdots + e_{KK})\sigma^{-1}(f_{11}) = \sigma^{-1}(f_{11}).
\]

So, applying \( \sigma \), we have \( \sigma(e_{11}) + \cdots + \sigma(e_{KK})f_{11} = f_{11} \). Multiply on the left to get

\[
[f_{11}\sigma(e_{11}) + \cdots + f_{11}\sigma(e_{KK})]f_{11} = f_{11}.
\]

Now evaluate at \( v_1 \) to get

\[
(1) \quad [f_{11}\sigma(e_{11}) + \cdots + f_{11}\sigma(e_{KK})]v_1 = v_1.
\]

Now let \( G = \sigma(e_{11})V \oplus \cdots \oplus \sigma(e_{KK})V \) (external direct sum), and consider the map:

\[
f: G \to v_1S \text{ given by } \quad (\sigma(e_{11})v_1, \ldots, \sigma(e_{KK})v_K) \to f_{11}(\sigma(e_{11})v_1 + \cdots + \sigma(e_{KK})v_K).
\]

Then (1) shows that \( f \) is an epimorphism. Since \( v_1S \approx S \), \( f \) splits, and since \( \sigma(e_{11})V \approx \sigma(e_{11})V \), we have \( [\sigma(e_{11})V]^\times \approx S \oplus C \), so \( \sigma(e_{11})V \) is a generator.

End \( \sigma(e_{11})V \approx R \). Let \( M \) be any module and \( M = N \oplus C \). Let \( e \) be the projection onto \( M \) along \( C \). Then it is well known and easy to prove that \( \text{End } N \approx e(\text{End } M)e \), so since \( V = \sigma(e_{11})(V) \oplus C \), \( \text{End } \sigma(e_{11})V = \text{End } \sigma(e_{11})(\text{End } V)\sigma(e_{11}) \). Apply \( \sigma^{-1} \):

\[
\text{End } \sigma(e_{11})V \approx e_{11}\sigma^{-1}(\text{End } V)e_{11} = e_{11}R Ne_{11} \approx R.
\]

REFERENCES


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